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Conservativeness and recurrence for generalized Dirichlet forms

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Conservativeness and recurrence for generalized Dirichlet forms

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Abstract

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In the thesis, we develop analytic criteria for recurrence, transience and conservativeness of non-sectorial perturbations of possibly non-symmetric Dirichlet forms on a metric measure space. These form an important subclass of generalized Dirichlet forms which were introduced in [36]. In case there exists an associated strong Feller process, the analytic conditions imply recurrence, transience and conservativeness, i.e. non-explosion of the associated process, in the classical probabilistic sense. As an application of our general results, we consider a generalized Dirichlet form given on a closed or open subset of $\mathbb{R}^d$ which is given as a divergence free first order perturbation of a symmetric energy form or a non-symmetric sectorial energy form. Then using volume growth conditions of the carré du champ and the non-sectorial first order part, we derive an explicit criterion for recurrence and conservativeness. We present concrete examples with applications to Muckenhoupt weights and counterexamples for recurrence. The counterexamples show that the non-sectorial case differs qualitatively from the symmetric or non-symmetric sectorial case. Namely, we make the observation that one of the main criteria for recurrence in these cases fails to be true for generalized Dirichlet forms. Moreover, we present several concrete examples for conservativeness which relate our results to previous ones obtained by different authors. In particular, we show that conservativeness can hold for a cubic variance if the drift is strong enough to compensate it.

Keywords: generalized Dirichlet forms, non-symmetric Dirichlet forms, recurrence, transience, conservativeness, non-explosion, Markov semigroups, Diffusion processes.

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Contents

Chapter 1 Introduction

This thesis is based on [9, 10, 11]. Recurrence, transience and conservativeness criteria for C_0-semigroups of contractions, non-explosion criteria for Markov processes and related problems are important topics both in analysis and probability theory. These were hence studied by many authors under various aspects (see for instance [1, 2, 4, 5, 6, 7, 12, 13, 14, 15, 16, 19, 21, 22, 23, 25, 27, 30, 32, 33, 36, 38, 39, 40] and references therein).

Here, we take a rather analytic point of view which fits to the frame of possibly unbounded and discontinuous coefficients. The main purpose of this thesis is to develop recurrence, transience and conservativeness criteria for (Markov processes $\mathbb{M}$ corresponding to) a generalized Dirichlet form which can be expressed as a linear perturbation of a sectorial Dirichlet form. This thesis consist of two parts.

In Part I, we develop sufficient analytic conditions for recurrence and transience and derive an explicit condition for recurrence. More precisely, we consider a locally compact separable metric space (E, d) with a locally finite (i.e. finite on compacts) positive measure μ with full support on E and a generalized Dirichlet form $\mathcal{E}$ that can be decomposed as

$$\mathcal{E}(u,v) = \mathcal{E}^0(u,v) + \int_E uNv\,d\mu, \tag{1.1}$$

where $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a sectorial Dirichlet form on $L^2(E,\mu)$ that is dominated by $\mathcal{E}$ on a subspace of the diagonal of $D(\mathcal{E}^0)$ and $(N, D(N))$ is a linear operator on $L^2(E,\mu)$. The precise conditions are formulated in (R1) and (R2) of Chapter

3 below.

Here as a warning, we emphasize that we use the term "sectorial" exclusively in the sense of strong sector condition (cf. Remark 3.2(i) and end of Remark 5(ii)) in Part I.

The class of generalized Dirichlet forms as in (1.1) is quite large. It contains symmetric Dirichlet forms as in [5], Dirichlet forms satisfying strong sector condition as in [18] (and also [23], if the dual semigroup is supposed to be sub-Markovian there) and time-dependent Dirichlet forms as in [24]. After having introduced the basic notions, for even more general forms as in (1.1), namely generalized Dirichlet forms satisfying (R1) and (R2), we derive some domination principle on the diagonal (see Theorem 3.1 and Remark 3.2) and the existence of a nice reference function in case of transience (see Lemma 3.1). Our main result for general forms as in (1.1) is Theorem 3.2 and Corollary 3.1 which constitute a generalization of the symmetric case of [5] and of the sectorial case of [23], if $(\widehat{T}_t)_{t>0}$ is sub-Markovian there (cf. Remark 3.3).

Recurrence and transience are described through potential operators and the potential operators can be defined in an analytic way through an underlying C_0-semigroup of contractions as for instance in (3.1) below or in a probabilistic way where the potential operator is defined through an underlying Markov process $\mathbb{M}$ as at the beginning of Section 3.2. In Section 3.2, we follow the main lines of the well-known work [12] to point out the connection of the analytic recurrence and transience to the probabilistic one. In particular, if the generalized Dirichlet form in (1.1) is associated to a right process $\mathbb{M}$ as at the beginning of Section 3.2, i.e. if

$$G_\alpha f = E_\cdot\left[\int_0^\infty e^{-\alpha t} f(X_t)dt\right] \ \mu\text{-a.e.}$$

for any bounded $f \in L^2(E,\mu)$ and $\alpha > 0$, then the analytic recurrence (resp. transience) of $\mathcal{E}$ can be described probabilistically as in Proposition 3.2 (resp. Proposition 3.1). Moreover, if the transition function $(p_t)_{t>0}$ of $\mathbb{M}$ is strong Feller, then the μ-a.e. statements of Propositions 3.1 and 3.2 can be transformed into everywhere statements as explained at the end of Section 3.2. Thus, we obtain pointwise recurrence as in the case of (Hölder) continuous or locally bounded coefficients (cf. for instance [1], [27]) even though in our situation the coefficients may be discontinuous and unbounded. In general, only the transition from μ-a.e. to $\mathcal{E}$-quasi-everywhere statements is possible in Propositions 3.1 and 3.2 through standard Dirichlet form theory arguments.

In Chapter 4, as an application of main result, we consider an open or closed subset E of $\mathbb{R}^d$ and adapting the arguments of [36] in particular to the case with reflection (cf. Lemma 4.1 and its proof in Chapter 5), we construct a generalized Dirichlet form on $L^2(E,\mu)$, $d\mu = \varphi dx$, $\varphi > 0$ dx-a.e., that extends

$$\mathcal{E}(f,g) = \int_E \langle A\nabla f, \nabla g\rangle d\mu - \int_E \langle B, \nabla f\rangle g d\mu, \tag{1.2}$$

where $A = (a_{ij})_{1\le i,j\le d}$ is a possibly non-symmetric matrix of locally μ-integrable functions and $B := (B_1,\dots,B_d) \in L^2_{loc}(E,\mathbb{R}^d,\mu)$ is μ-divergence free (see (4.3) below). For the precise conditions, we refer to Chapter 4. In particular, we show that the form (1.2) fits into the frame of (1.1) and we obtain first sufficient recurrence and transience criteria for (1.2) by applying the results of Section 3.1 (cf. Corollary 4.1 and Remark 4.2). Then following a construction scheme of [5] that we adapt to the non-sectorial case (cf. Lemmas 4.2 and 4.3), we show that recurrence of $\mathcal{E}$ in (1.2) implies recurrence of its symmetric part (cf. Theorem 4.1) and conservativeness of $\mathcal{E}$ (cf. Corollary 4.2). For ease of exposition, some proofs of Section 4.1 are postponed to Chapter 5.

In Section 4.1, we derive explicit conditions for recurrence under the existence of a function ρ (see beginning of Section 4.1) which always exists if E is closed and so in particular if $E = \mathbb{R}^d$. Our main result here is Theorem 4.2 that characterizes recurrence in terms of volume growth. It can be seen as a generalization of [33, Theorem 3] in the Euclidean case.
In Section 4.2, we present examples and counterexamples. The counterexamples show that the non-sectorial case differs from the symmetric and from the non-symmetric sectorial case. In order to explain the difference, we first recall the well-known sufficient conditions for recurrence in the sectorial case:
If $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a symmetric Dirichlet form on $L^2(E,\mu)$, then the existence of $(\chi_n)_{n\ge1} \subset D(\mathcal{E}^0)$ such that $0 \le \chi_n \le 1$, $\lim_{n\to\infty} \chi_n = 1$ μ-a.e. and

$$\lim_{n\to\infty} \mathcal{E}^0(\chi_n, \chi_n) = 0$$

is an equivalent condition for (analytic) recurrence of $(\mathcal{E}^0, D(\mathcal{E}^0))$ (see [5, Theorem 1.6.3] and beginning of Section 4.1). In addition, if $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a sectorial Dirichlet form and strictly irreducible, then the existence of $(\chi_n)_{n\ge1} \subset D(\mathcal{E}^0)$ such that $0 \le \chi_n \le 1$, $\lim_{n\to\infty} \chi_n = 1$ μ-a.e. and $\lim_{n\to\infty} \mathcal{E}^0(\chi_n, \chi_n) = 0$ is a sufficient condition for recurrence of $(\mathcal{E}^0, D(\mathcal{E}^0))$ (see [23, Theorem 1.3.9]). In Subsections 4.2.1 and 4.2.2, we present several counterexamples of generalized Dirichlet forms as in (1.2) for which there exists $(\chi_n)_{n\ge1}$ as above with

$$\lim_{n\to\infty} \mathcal{E}(\chi_n, \chi_n) = 0,$$

but $\mathcal{E}$ is not recurrent. In Subsection 4.2.3, we discuss concrete examples in the case where the density φ is in some Muckenhoupt class.
In Section 4.3, as an application of the criterion for recurrence of the symmetric strongly local Dirichlet forms, we present explicit sufficient conditions for

symmetric gradient type Dirichlet forms satisfying Hamza type condition on $\mathbb{R}$ to be recurrent.

Chapter 5 is as already mentioned devoted to the postponed proofs of Chapter 4.

In Part II, we develop sufficient analytic conditions for conservativeness. We consider a locally compact separable metric space (E,d), a locally finite positive measure μ with full support on E and a generalized Dirichlet form $\mathcal{E}$ that can be decomposed locally as

$$\mathcal{E}(u,v) = \mathcal{E}^0(u,v) + \int_E uNv d\mu,$$

where $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a symmetric strongly local Dirichlet form on $L^2(E,\mu)$ represented by a carré du champ Γ and $(N, D(N))$ is a linear operator on $L^2(E,\mu)$. The precise conditions are formulated in localized form as (C1), (C2) in Chapter 6 below. We further assume (C3), (C4) which are also formulated in Chapter 6. (C3) corresponds to [33, Assumption (A)]) and its consequence [33, Lemma 1], i.e. (C3) allows us to obtain nice cut-off functions (see (6.11)) and to obtain a suitable exhaustive sequence for the state space (see (6.7)). In Remark 6.1, we explain why any symmetric strongly local and regular Dirichlet form satisfying [33, Assumption (A)]) satisfies (C1)-(C3). Since the semigroups that we consider are in general not analytic, we have to impose the denseness condition (C4), where the set D_0 that occurs in (C4) is given as in (6.9). Remark 6.2 explains more on D_0, (C4) and condition (A) that is just used as an auxiliary assumption to perform further calculations (see the sentence right before condition (A)). In Lemma 6.1, we include for the reader's convenience a proof to the fact that the conservativeness of the semigroup $(T_t)_{t>0}$ on

$L^\infty(E,\mu)$ (obtained from the $L^2(E,\mu)$-semigroup associated to $\mathcal{E}$) is equivalent to the $(\widehat{T}_t)_{t>0}$-invariance on $L^1(E,\mu)$. In Lemma 6.2, we derive similarly to [25] an equivalent criterion for the $(\widehat{T}_t)_{t>0}$-invariance in localized form. In order to estimate the limit in Lemma 6.2 by the Davies method, we use the functions ψ_n defined in (6.12) via the function ϕ defined right before display (6.12) and then define the "Davies semigroup" in (6.13). Then in a series of calculations, starting from (6.14), and using the key inequality (6.18) which only holds for divergence free perturbations, i.e. because of (6.5), we obtain our main Theorem 6.1 and its Corollary 6.1. Theorem 6.1 and Corollary 6.1 form the core of Part II and will be used to obtain explicit conservativeness criteria in the symmetric, non-symmetric and non-sectorial case.

The organization of the following Chapters 7 and 8 are then as follows. In Chapter 7, we consider applications of our core results to the symmetric case. Here our results are comparable to [25] (see Example 7.1 and Remark 7.1) and we recover a result of [33] (see Remark 7.1 and also [13] and [39] and references therein) by applying our main Proposition 7.1. In Section 7.2, we consider sectorial perturbations of symmetric Dirichlet forms and we are able to reconfirm a result on conservativeness from [29, Lemma 5.4] in Subsection 7.2.1. In Section 7.3, we show that Theorem 6.1 and Corollary 6.1 are also applicable to non-symmetric Dirichlet forms with non-symmetric diffusion matrix. The key observation is that the anti-symmetric part of the diffusion matrix becomes a μ-divergence free vector field after integration by parts. The sufficient criteria (7.12) and (7.13) for conservativeness extend the result of [40] in the sense that we can also consider invariant measures $\mu = \varphi^2 dx$ where $\varphi \not\equiv 1$. We show that we can also recover the result of [40] to some extend in case $\varphi \equiv 1$ in Subsection

7.2.1.

In Chapter 8, we consider non-sectorial perturbations of symmetric Dirichlet forms on Euclidean space as introduced in Chapter 4 and [10]. For the convenience of the reader, we explain in concise form the construction of the underlying generalized Dirichlet form, how the constructed generalized Dirichlet forms fits into the frame of Chapter 6, as well as some of its main properties. Subsequently, we apply the conservativeness criterion of Chapter 6 to formulate Corollary 8.1 and to obtain two different explicit examples. The first example of Subsection 8.2.1 shows that conservativeness can hold for a cubic variance if the drift is strong enough to compensate it. The second example of Subsection 8.2.2 indicates that our conservativeness criteria in dimension one can be in some situations sharper than the ones of [33], but not as sharp as the Feller test is (cf. Remark 8.1).

Let us finally explain our main motivation for this work. Conservativeness criteria lead to uniqueness results both at analytic and probabilistic level. Let us discuss both of these. The non-symmetry assumption (or even the lack of sector condition) is here of particular importance, since it leads to a wider class of semigroups and stochastic processes to which the conservativeness criteria can be applied than the restrictive assumption of symmetry. It is pointed out in [36] that the $(\widehat{T}_t)_{t>0}$-invariance of the underlying measure μ is related to the L^1-uniqueness of the corresponding infinitesimal generator and can be applied to obtain existence of a unique invariant measure. On the other hand $(\widehat{T}_t)_{t>0}$-invariance is equivalent to the conservativeness of the dual semigroup $(T_t)_{t>0}$ (cf. Lemma 6.1). Thus, conservativeness criteria can be used to obtain L^1-uniqueness and existence of unique invariant measures for Markov semigroups.

The second important application of the conservativeness criteria that we study is the relation to new non-explosion results for solutions to singular SDE which were constructed probabilistically up to an explosion time in [17] and [45]. There it is shown that certain SDE in $\mathbb{R}^d$ with merely L^p-integrability conditions on the dispersion and drift coefficients have pathwise unique and strong solutions up to their explosion times, i.e. the random times at which they leave $\mathbb{R}^d$. Thus, if we can construct weak solutions to these SDE via (generalized or non-symmetric) Dirichlet form theory, then the analytic conservativeness criteria lead to new non-explosion results for these SDE. We refer the interested reader to the articles [29], [31] where this kind of application has been studied and to Subsection 7.2.1 where the results of this article are applied to obtain a considerably shorter proof for conservativeness than in [29, Lemma 5.4]. For further related work in the context of applications that we are interested in, we refer to the recent work [44] where non-explosion and existence and uniqueness of invariant measures is investigated.

Chapter 2 Framework

Let us introduce the framework which is kept throughout both Part I and Part II.

Let (E,d) be a locally compact separable metric space and let μ be a locally finite (i.e. finite on compacts) positive measure on its Borel σ-algebra $\mathcal{B}(E)$. We assume that μ has full support. The closure of $A \subset E$ will be denoted by $\overline{A}$ and $A^c := E \setminus A$ stands for the complement of A in E. For $1 \leq p < \infty$, let $L^p(E,\mu)$ be the space of equivalence classes of p-integrable functions with respect to μ and $L^\infty(E,\mu)$ be the space of μ-essentially bounded functions. We denote the corresponding norms by $\|\cdot\|_{L^p(E,\mu)}$, $p \in [1,\infty]$ and to make notations easier, we do not distinguish at times between equivalence class and representative. The inner product of the Hilbert space $\mathcal{H} := L^2(E,\mu)$ will be denoted by $(\ ,\ )$.

The support of a function u on E (=support of $|u|d\mu$) is denoted by $\text{supp}(u)$. For any set of functions W on E, we will denote by W_0 the set of functions $u \in W$ which have a compact support in E and by W_b the set of functions in W which are bounded μ-a.e. and let W_{loc} be the set of measurable functions u such that for any relatively compact open set V, there exists $v \in W$ with $u = v$ μ-a.e. on V. Let $W_{0,b} := W_0 \cap W_b$ and define $W_{loc,b}$ by the set of bounded measurable functions u such that $u \in W_{loc}$. Let $C_0(E)$ be the set of continuous functions u such that $\text{supp}(u)$ is a compact in E and $C_b(E)$ be the set of bounded continuous functions. We say that a statement holds for $n \gg 1$, if there exists some $N \in \mathbb{N}$ such that the statement holds for any $n \geq N$.

Let $(\mathcal{A},\mathcal{V})$ be a Dirichlet form (not necessarily symmetric) on $L^2(E,\mu)$ in the sense of [18, I. Definition 4.5]. So $\mathcal{V}$ is a real Hilbert space with respect to the norm $\|u\|_{\mathcal{V}}^2 := \mathcal{A}_1(u,u) := \mathcal{A}(u,u)+(u,u)$. Denote the dual space of $\mathcal{V}$ by $\mathcal{V}'$. Assume that there exists a linear operator $(\Lambda, D(\Lambda,\mathcal{H}))$ on $L^2(E,\mu)$ which is a generator of a sub-Markovian C_0-semigroup of contractions $(U_t)_{t>0}$ on $L^2(E,\mu)$ that can be restricted to a C_0-semigroup on $\mathcal{V}$. Then the conditions (D1) and (D2) in [37, Chapter I] are satisfied. In particular, $\Lambda : D(\Lambda,\mathcal{H})\cap\mathcal{V}\longrightarrow\mathcal{V}'$ is closable. Let $(\Lambda,\mathcal{F})$ be the closure of $\Lambda : D(\Lambda,\mathcal{H})\cap\mathcal{V}\longrightarrow\mathcal{V}'$. Then $\mathcal{F}$ is a real Hilbert space with corresponding norm

$$\|u\|_{\mathcal{F}}^2 := \|u\|_{\mathcal{V}}^2 + \|\Lambda u\|_{\mathcal{V}'}^2.$$

Let $\mathcal{E}$ be the bilinear form associated with $(\mathcal{A},\mathcal{V})$ and $(\Lambda, D(\Lambda,\mathcal{H}))$ (see [37, I. Definition 2.9]). Then $\mathcal{E}$ is a generalized Dirichlet form (see [37, I. Proposition 4.7]). In particular, for $u\in\mathcal{F},\ v\in\mathcal{V}$, $\mathcal{E}$ can be written as

$$\mathcal{E}(u,v) = \mathcal{A}(u,v) - {}_{\mathcal{V}'}\langle\Lambda u, v\rangle_{\mathcal{V}}.$$

Let $(G_\alpha)_{\alpha>0}$ and $(\widehat{G}_\alpha)_{\alpha>0}$ on $L^2(E,\mu)$ be associated with $\mathcal{E}$, i.e. $(G_\alpha)_{\alpha>0}$ is the sub-Markovian C_0-resolvent of contractions on $L^2(E,\mu)$ satisfying $G_\alpha(\mathcal{H})\subset\mathcal{F}$,

$$\mathcal{E}_\alpha(G_\alpha f, g) = (f,g),\quad f\in L^2(E,\mu),\ g\in\mathcal{V},$$

where $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v)+\alpha(u,v)$ for $\alpha>0$ and $(\widehat{G}_\alpha)_{\alpha>0}$ is the adjoint C_0-resolvent of contractions of $(G_\alpha)_{\alpha>0}$ (see [37, I. Proposition 3.6]). By [18, I. Proposition 1.5], there exists exactly one linear operator $(L,D(L))$ (resp. $(\widehat{L},D(\widehat{L}))$) on $L^2(E,\mu)$ corresponding to $(G_\alpha)_{\alpha>0}$ (resp. $(\widehat{G}_\alpha)_{\alpha>0}$). Then $(\widehat{L},D(\widehat{L}))$ is the adjoint operator of $(L,D(L))$. Let $(T_t)_{t>0}$ and $(\widehat{T}_t)_{t>0}$ be the C_0-semigroups of contractions corresponding to $(G_\alpha)_{\alpha>0}$ and $(\widehat{G}_\alpha)_{\alpha>0}$ respectively. $(\widehat{T}_t)_{t>0}$ restricted to $L^1(E,\mu)\cap L^2(E,\mu)$ can be extended to a C_0-semigroup of contractions on $L^1(E,\mu)$. This extension will also be denoted by $(\widehat{T}_t)_{t>0}$. $(\widehat{T}_t)_{t>0}$ is not necessarily sub-Markovian, however from (R1) in Part I and (C1) in Part II on (see below), the sub-Markovianity of $(\widehat{T}_t)_{t>0}$ follows and is hence assumed to hold.

Part I

Recurrence criteria for generalized Dirichlet forms

Chapter 3 Analytic and probabilistic characterization of recurrence and transience

This Chapter consists of two parts. In the first part, we characterize recurrence and transience analytically in the non-sectorial case and derive an analytic criterion for a generalized Dirichlet form to be recurrent or more generally non-transient. In the second part, we show that the analytic characterization of recurrence and transience indeed implies recurrence and transience in the classical probabilistic sense in case there exists a process associated with the generalized Dirichlet form.

3.1 A general criterion for recurrence and transience of a generalized Dirichlet form

We develop sufficient analytic conditions and criteria for recurrence and transience of $\mathcal{E}$ which is decomposed as

$$\mathcal{E}(u,v) = \mathcal{E}^0(u,v) + \int_E uNv d\mu,$$

where $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a sectorial Dirichlet form (a Dirichlet form satisfying strong sector condition) on $L^2(E,\mu)$ that is dominated by $\mathcal{E}$ on a subspace of the diagonal of $D(\mathcal{E}^0)$ and $(N, D(N))$ is a linear operator on $L^2(E,\mu)$ (see (R2) below). From now on until the end of Section 3.1, we assume:

(R1) $(\widehat{T}_t)_{t>0}$ is sub-Markovian.

Then $(T_t)_{t>0}$ restricted to $L^1(E,\mu) \cap L^2(E,\mu)$ can be extended to a semigroup of contractions on $L^1(E,\mu)$, which is actually equivalent to (R1). Since $(T_t)_{t>0}$ is positivity preserving, so is its $L^1(E,\mu)$-version. Let $f \in L^1(E,\mu)$ with $f \geq 0$. Then for $0 \leq N \leq M$,

$$0 \leq \int_0^N T_t f dt \leq \int_0^M T_t f dt$$

and for $0 \leq \beta \leq \alpha$,

$$0 \leq \int_0^\infty e^{-\alpha t} T_t f dt \leq \int_0^\infty e^{-\beta t} T_t f dt.$$

Hence,

$$Gf := \lim_{N\to\infty} \int_0^N T_t f dt = \lim_{\alpha\to 0} \int_0^\infty e^{-\alpha t} T_t f dt \leq \infty \tag{3.1}$$

is uniquely defined μ-a.e. G is called potential operator associated with $(T_t)_{t>0}$.

DEFINITION **3.1**

(i) $(T_t)_{t>0}$ *is said to be recurrent, if for any* $f \in L^1(E,\mu)$ *with* $f \geq 0$ *μ-a.e., we have*

$$Gf = 0 \quad or \quad \infty \ \ \mu\text{-}a.e.$$

(ii) $(T_t)_{t>0}$ *is said to be transient, if there exists* $g \in L^1(E,\mu)$ *with* $g > 0$ *μ-a.e. such that*

$$Gg < \infty \ \ \mu\text{-}a.e.$$

(iii) Likewise, we can define recurrence and transience of any operator which is a generator of a positivity preserving semigroup of contractions on $L^1(E,\mu)$.

DEFINITION **3.2**

(i) *A measurable set* $B \in \mathcal{B}(E)$ *is called weakly invariant set relative to* $(T_t)_{t>0}$*, if*

$$T_t(f \cdot 1_B) = 0 \ \mu\text{-a.e. on } B^c$$

for any $t > 0$, $f \in L^2(E,\mu)$.

(ii) $(T_t)_{t>0}$ *is said to be strictly irreducible, if for any weakly invariant set* B *relative to* $(T_t)_{t>0}$*, we have*

$$\mu(B) = 0 \text{ or } \mu(B^c) = 0.$$

REMARK **3.1** *From [16, Section 2], we deduce:*

(i) $(T_t)_{t>0}$ *is transient, if and only if* $Gf < \infty$ μ*-a.e. for any* $f \in L^1(E,\mu)$ *with* $f \geq 0$ μ*-a.e.*

(ii) *If* $g \in L^1(E,\mu)$ *with* $g > 0$ μ*-a.e., then* $\{x \in E : Gg(x) = \infty\}$ *is a weakly invariant set relative to* $(T_t)_{t>0}$*. Consequently, if* $(T_t)_{t>0}$ *is strictly irreducible, then it is either recurrent or transient.*

(iii) *If there exists a strictly positive measurable function* $(p_t(x,y))_{t>0,x,y\in E}$ *with*

$$T_t f(x) = \int_E p_t(x,y) f(y) \mu(dy)$$

for any $x \in E$, $t > 0$ *and* $f \in L^2(E,\mu)$*, then* $(T_t)_{t>0}$ *is strictly irreducible.*

(iv) *In the symmetric case (cf. [5]),* $B \in \mathcal{B}(E)$ *is weakly invariant, if and only if it is invariant in the sense of [5, Chapter 1.1.6]. Therefore, a symmetric Dirichlet form is irreducible if and only if it is strictly irreducible.*

Now, we shall show that the transience of $(T_t)_{t>0}$ is determined by the symmetric part of the corresponding generalized Dirichlet form under some domination on the diagonal.

THEOREM **3.1** *If there exists a sectorial Dirichlet form* $(\mathcal{E}^0, D(\mathcal{E}^0))$ *in the sense of [18, I. Definition 4.5 and I. (2.4)] such that* $D(L)_b \subset D(\mathcal{E}^0)$ *and*

$$\mathcal{E}^0(u,u) \leq \mathcal{E}(u,u)$$

for any $u \in D(L)_b$*, then the transience of* $(\mathcal{E}^0, D(\mathcal{E}^0))$ *implies the transience of* $(T_t)_{t>0}$.

Proof If $(\mathcal{E}^0, D(\mathcal{E}^0))$ is transient, then by Remark 3.1(i), there exists $g^0 \in L^1(E,\mu)_b$ with $g^0 > 0$ μ-a.e. such that

$$G^0 g^0 < \infty \ \ \mu\text{-a.e.}$$

where G^0 is the potential operator associated with $(\mathcal{E}^0, D(\mathcal{E}^0))$. Let

$$g := \frac{g^0}{\max(G^0 g^0, 1)}.$$

Then, $g \in L^1(E,\mu)_b$ satisfies $\int_E g G^0 g d\mu < \infty$. According to [23, Theorem 1.3.9], there exists a constant $K_g > 0$ depending on g and the sector constant of $(\mathcal{E}^0, D(\mathcal{E}^0))$ such that

$$\int_E |u| g d\mu \leq K_g \mathcal{E}^0(u,u)^{1/2} \tag{3.2}$$

for any $u \in D(\mathcal{E}^0)$. Choosing u to be $G_\alpha g \in D(L)_b$ in (3.2) where $\alpha > 0$, we get

$$\int_E g G_\alpha g d\mu \leq K_g \mathcal{E}^0(G_\alpha g, G_\alpha g)^{1/2} \leq K_g \mathcal{E}(G_\alpha g, G_\alpha g)^{1/2} \leq K_g \int_E g G_\alpha g d\mu^{1/2}.$$

Therefore, $\int_E gG_\alpha g d\mu \leq K_g^2$ holds for any $\alpha > 0$, and it follows by B. Levi that

$$\int_E gGgd\mu \leq K_g^2.$$

Consequently, we obtain $Gg < \infty$ μ-a.e.

$\square$

REMARK **3.2**

(i) In Part I, the term "sectorial" is exclusively meant in the sense of satisfying strong sector condition (2.4) of [18, Chapter I. 2].

(ii) Let $(\mathcal{E}^0, D(\mathcal{E}^0))$ be a sectorial Dirichlet form on $\mathcal{H}$. Then its symmetric part $(\widetilde{\mathcal{E}}^0, D(\mathcal{E}^0))$ is a symmetric Dirichlet form on $\mathcal{H}$ (see [18, I. Exercise 4.6]). By Theorem 3.1, we obtain: a sectorial Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ is transient, if and only if $(\widetilde{\mathcal{E}}^0, D(\mathcal{E}^0))$ is transient. Note that if $(\mathcal{E}^0, D(\mathcal{E}^0))$ is only assumed to satisfy the weak sector condition as in [18, (2.3) of Chapter I], then its symmetric part may be recurrent, while $(\mathcal{E}^0, D(\mathcal{E}^0))$ is not. This can be seen in Example 4.1 below.

Since $(T_t)_{t>0}$ is a sub-Markovian C_0-semigroup of contractions on $L^2(E, \mu)$, $(T_t)_{t>0}$ restricted to $L^2(E, \mu) \cap L^\infty(E, \mu)$ can be extended to a linear operator on $L^\infty(E, \mu)$. In fact, for $f \in L^\infty(E, \mu)$ with $f \geq 0$ μ-a.e., we set

$$T_t f := \lim_{n \to \infty} T_t f_n$$

where $f_n \in L^2(E, \mu) \cap L^\infty(E, \mu)$ such that $f_n \nearrow f$ μ-a.e. as $n \to \infty$. Since $(T_t)_{t>0}$ is positivity preserving, the limit is well-defined μ-a.e. and is independent of the choice of approximating sequence $(f_n)_{n\geq 1}$. For general $f \in L^\infty(E,\mu)$, considering the decomposition $f = f^+ - f^-$ in positive and negative parts, $T_t f$ is well-defined by $T_t f := T_t f^+ - T_t f^-$. Furthermore for any $t, s > 0$,

$$T_t T_s f = T_t\left(\lim_{n\to\infty} T_s f_n\right) = \lim_{n\to\infty} T_t T_s f_n = \lim_{n\to\infty} T_{t+s} f_n = T_{t+s} f, \ \ \mu\text{-a.e.}$$

Consequently, $(T_t)_{t>0}$ can be considered as a sub-Markovian semigroup of contractions on $L^\infty(E,\mu)$. The potential operator G relative to $(T_t)_{t>0}$ can be regarded as an operator on $L^\infty(E,\mu)$.

Using an idea from [32, Theorem 15] about invariant sets of discrete semigroups in the proof of the next lemma, we show that g and Gg in Definition 3.1(ii) can be chosen μ-uniformly bounded.

LEMMA **3.1** *If $(T_t)_{t>0}$ is transient, then there exists a function $g \in L^1(E,\mu)_b$ with $g > 0$ μ-a.e. and $Gg \in L^\infty(E,\mu)$.*

Proof Fix $f \in L^1(E,\mu)_b$ with $f > 0$ μ-a.e.,

$$\|f\|_{L^\infty(E,\mu)} \leq 1 \text{ and } \|f\|_{L^1(E,\mu)} \leq 1.$$

By Remark 3.1(i), we have

$$0 < Gf < \infty \ \ \mu\text{-a.e.}$$

Define functions for $m, k \geq 1$, by

$$g_{mk} := Gf \wedge m - T_k(Gf \wedge m)$$

where $a \wedge b := \min\{a, b\}$. Then $\|g_{mk}\|_{L^\infty(E,\mu)} \leq m$. Moreover, if $x \in E$ is such that $Gf(x) < m$, then since T_k is positivity preserving,

$$\begin{aligned} g_{mk}(x) &= Gf(x) - T_k(Gf \wedge m)(x) \\ &\geq Gf(x) - T_k(Gf)(x) = \int_0^k T_t f(x)dt \geq 0. \end{aligned}$$

If $x \in E$ is such that $Gf(x) \geq m$, then since T_k is sub-Markovian, we have

$$g_{mk}(x) = m - T_k(Gf \wedge m)(x) \geq 0.$$

Consequently, $g_{mk} \geq 0$ μ-a.e. Define for $m, k \geq 1$,

$$A_m := \{x \in E : (m-1) \leq Gf(x) < m\}$$

and

$$B_k := \left\{x \in E : \int_0^{k-1} T_t f(x) dt = 0 \text{ but } \int_0^k T_t f(x) dt > 0\right\}.$$

Since $(T_t)_{t>0}$ is transient, $\bigcup_{m=1}^{\infty} A_m = \bigcup_{k=1}^{\infty} B_k = E$ up to some μ-negligible set. Without loss of generality, we may assume that $\mu(A_m \cap B_k) < \infty$ for any $m, k \geq 1$. Otherwise, we may subdivide $A_m \cap B_k$ in countably many pairwise disjoint sets with finite μ-measure and proceed as below. Let $c_{mk} := \max(1, \mu(A_m \cap B_k))$ and

$$\widetilde{g}_{mk} := g_{mk} \cdot 1_{A_m \cap B_k}.$$

Then, we obtain that

$$\|\widetilde{g}_{mk}\|_{L^\infty(E,\mu)} \leq m, \quad \|\widetilde{g}_{mk}\|_{L^1(E,\mu)} \leq m \cdot \mu(A_m \cap B_k), \quad \widetilde{g}_{mk} > 0 \text{ on } A_m \cap B_k$$

and

$$G\widetilde{g}_{mk} \leq Gg_{mk} = G\left(Gf \wedge m - T_k(Gf \wedge m)\right) = \int_0^k T_t(Gf \wedge m) dt \leq mk$$

μ-a.e. on E. Consequently,

$$g := \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{\widetilde{g}_{mk}}{2^m 2^k c_{mk}}$$

satisfies the desired properties.

□

Next, we give a general criterion for recurrence in case the generalized Dirichlet form can be represented by a linear perturbation of a sectorial Dirichlet form. By this, we mean that there exist a sectorial Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ in the sense of [18, I. Definition 4.5 and I. (2.4)] with $D(L)_b \subset D(\mathcal{E}^0)$ and a linear operator $(N, D(N))$ on $\mathcal{H}$ such that

$$\mathcal{E}(u,v) = (-Lu, v) = \mathcal{E}^0(u,v) + \int_E uNv d\mu \tag{3.3}$$

for any $u \in D(L)_b$ and $v \in D(N) \cap D(\mathcal{E}^0)$. Note that the linear operator $(N, D(N))$ needs not to be a generator of a C_0-semigroup of contractions on $\mathcal{H}$. Thus from now on, we assume that the generalized Dirichlet form $\mathcal{E}$ satisfies the following condition:

(R2) $\mathcal{E}$ can be decomposed as in (3.3) and

$$\mathcal{E}^0(u,u) \le \mathcal{E}(u,u)$$

for any $u \in D(L)_b$.

Let

$$D := \left\{ u \in D(N) \cap D(\mathcal{E}^0) : Nu \in L^1(E, \mu) \right\}.$$

For the given sectorial Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$, we define the *extended Dirichlet space* of $D(\mathcal{E}^0)$ as the set of all measurable functions u with $|u| < \infty$ μ-a.e. for which there exists an $\mathcal{E}^0$-Cauchy sequence $(u_n)_{n \ge 1} \subset D(\mathcal{E}^0)$ such that

$$\lim_{n\to\infty} u_n = u \ \mu\text{-a.e.}$$

(see [23, Chapter 1.3]). Since the Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ is sectorial, for u in the extended Dirichlet space,

$$\mathcal{E}^0(u,u) := \lim_{n\to\infty} \mathcal{E}^0(u_n, u_n)$$

exists and is independent of the choice of $(u_n)_{n\geq 1} \subset D(\mathcal{E}^0)$ (this can be shown as in the paragraph right before [23, Theorem 1.3.9]).

THEOREM **3.2** *Suppose $(T_t)_{t>0}$ is transient and let g be as in Lemma 3.1. Then Gg is in the extended Dirichlet space of $D(\mathcal{E}^0)$ and one can define*

$$\mathcal{E}^0(Gg, u) := \lim_{\alpha\to 0} \mathcal{E}^0(G_\alpha g, u)$$

for $u \in D(\mathcal{E}^0)$. Moreover, if $u \in D$, then

$$(u, g) = \mathcal{E}^0(Gg, u) + \int_E Gg \cdot Nu d\mu. \tag{3.4}$$

Proof Suppose that $(T_t)_{t>0}$ is transient and let $g \in L^1(E, \mu)_b$ with $g > 0$ μ-a.e. such that $Gg \in L^\infty(E, \mu)$. Since for any $\alpha > 0$,

$$\mathcal{E}^0(G_\alpha g, G_\alpha g) \leq \mathcal{E}(G_\alpha g, G_\alpha g) = -\int_E LG_\alpha g G_\alpha g d\mu \leq \int_E gGg d\mu < \infty,$$

there exists an $\mathcal{E}^0$-Cauchy sequence $(g_n)_{n\geq 1} \subset D(\mathcal{E}^0)$ consisting of a Cesàro mean of $(G_{\alpha_n} g)_{n\geq 1}$ for $\alpha_n \to 0$ as $n \to \infty$. Indeed, this follows from the theorems of Banach/Alaoglu and Banach/Saks applied in the abstract completion of $(\mathcal{E}^0, D(\mathcal{E}^0))$. Consequently, $\mathcal{E}^0(Gg, Gg) = \lim_{n\to\infty} \mathcal{E}^0(g_n, g_n)$ exists and $\lim_{n\to\infty} g_n = Gg$ μ-a.e. On the other hand, by the special form of $(g_n)_{n\geq 1}$,

$$\mathcal{E}^0(Gg, Gg) \leq \int_E gGg d\mu \quad \text{and} \quad \lim_{n\to\infty} \mathcal{E}^0(g_n, u) = \lim_{n\to\infty} \mathcal{E}^0(G_{\alpha_n} g, u)$$

for any $u \in D(\mathcal{E}^0)$. By the assumption on $u \in D$, we have for any $n \geq 1$

$$(g, u) - \alpha_n(G_{\alpha_n} g, u) = (-LG_{\alpha_n} g, u) = \mathcal{E}^0(G_{\alpha_n} g, u) + \int_E G_{\alpha_n} g \cdot Nu d\mu.$$

Since $\lim_{n\to\infty} G_{\alpha_n} g = Gg$ μ-a.e. and $Nu \in L^1(E, \mu)$, we obtain by Lebesgue

$$\lim_{n\to\infty} \int_E G_{\alpha_n} g \cdot Nu d\mu = \int_E Gg \cdot Nu d\mu.$$

Since $\mathcal{E}_\alpha(G_\alpha g, G_\alpha g) = \int_E gG_\alpha g d\mu \le \int_E gGgd\mu$, we get for any $\alpha_n > 0$,

$$|\alpha_n(G_{\alpha_n} g, u)| \le \left(\alpha_n \int_E gGgd\mu\right)^{1/2} \cdot \|u\|_{L^2(E,\mu)}.$$

Let $n \to \infty$ (i.e. $\alpha_n \to 0$) then we obtain (3.4).

$\square$

COROLLARY **3.1**

(i) *If there exists a sequence of functions $(\chi_n)_{n\ge1} \subset D$ with $0 \le \chi_n \le 1$, $\lim_{n\to\infty} \chi_n = 1$ μ-a.e. satisfying*

$$\lim_{n\to\infty}\left(\mathcal{E}^0(g,\chi_n) + \int_E gN\chi_n d\mu\right) = 0, \tag{3.5}$$

for any non-negative bounded g in the extended Dirichlet space of $D(\mathcal{E}^0)$, then $(T_t)_{t>0}$ is not transient.

(ii) *If $(T_t)_{t>0}$ is strictly irreducible and there is a sequence of functions $(\chi_n)_{n\ge1} \subset D$ with $0 \le \chi_n \le 1$, $\lim_{n\to\infty} \chi_n = 1$ μ-a.e. satisfying (3.5), then $(T_t)_{t>0}$ is recurrent by Remark 3.1(ii).*

REMARK **3.3** *If $\mathcal{E}$ is a symmetric Dirichlet form, then we can drop the assumption that $(T_t)_{t>0}$ is strictly irreducible in Corollary 3.1(ii). Indeed, in this case one can use the (weak) invariance of $E_d := \{x \in E : Gg(x) < \infty\}$, $g \in L^1(E,\mu)$, $g > 0$ μ-a.e. and the reduced form on E_d in order to conclude (cf. proof of [5, Theorem 1.6.3]). Thus Corollary 3.1(ii) can be seen as generalization of the symmetric case [5, Theorem 1.6.5]. However, in our general* *non-symmetric situation, even if $N \equiv 0$, E_d is not weakly invariant in general and tools as in the symmetric case are not at hand. Consequently, strict irreducibility is imposed in Corollary 3.1(ii). Moreover, Theorem 3.2 is a generalization [23, Theorem 1.3.9] in case $(\widehat{T}_t)_{t>0}$ is sub-Markovian.*

3.2 Connection to recurrence and transience in the classical sense

For all notations, results that may not be defined, proved and cited in this Section, we refer to [5].

Let $\mathbb{M} = \big(\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta}\big)$ with life time ζ be a right process with state space E, resolvent $R_\alpha f(x) := E_x\left[\int_0^\infty e^{-\alpha t} f(X_t)dt\right]$ and semigroup $p_t f(x) := E_x\left[f(X_t)\right]$, $x \in E$, $\alpha > 0$, $t > 0$, $f \in B(E)_b$, where $B(E)$ denotes the set of Borel measurable functions on E, E_x denotes the expectation with respect to P_x. We assume that the measure μ is excessive relative to $(p_t)_{t>0}$, i.e.

$$\int_E p_t 1_A(x)\mu(dx) \leq \mu(A), \;\; A \in \mathcal{B}(E).$$

Hence, $(p_t)_{t>0}$ can be regarded as a linear operator sending a μ-equivalence class to another μ-equivalence class and can be extended as a linear operator on $L^1(E,\mu)$.

We are able to define recurrence and transience of $\mathbb{M}$ as in Definition 3.1. The Markov process $\mathbb{M}$ is said to be recurrent, if for any $f \in L^1(E,\mu)$ with $f \geq 0$ μ-a.e., we have

$$E_x\left[\int_0^\infty f(X_t)dt\right] = 0 \text{ or } \infty \;\; \mu\text{-a.e. } x \in E.$$

$\mathbb{M}$ is said to be transient, if there exists $g \in L^1(E,\mu)$ with $g > 0$ μ-a.e., such that

$$E_x\left[\int_0^\infty g(X_t)dt\right] < \infty \ \mu\text{-a.e. } x \in E.$$

$B \in \mathcal{B}(E)$ is said to be weakly invariant relative to $\mathbb{M}$, if for any $t > 0$,

$$E_x\left[1_B(X_t)\right] = 0 \ \mu\text{-a.e. } x \in B^c.$$

$\mathbb{M}$ is said to be strictly irreducible, if for any weakly invariant set B relative to $\mathbb{M}$, we have

$$\mu(B) = 0 \quad \text{or} \quad \mu(B^c) = 0.$$

A function u is said to be excessive, if $p_t u(x) \nearrow u(x)$ as $t \searrow 0$ for any $x \in E$. For $\omega \in \Omega$, define the first hitting time σ_B and the last exit time L_B from B by

$$\sigma_B(\omega) := \inf\{t > 0 : X_t(\omega) \in B\} \quad \text{and} \quad L_B(\omega) := \sup\{t \geq 0 : X_t(\omega) \in B\}.$$

Note that σ_B is the $\mathcal{F}_t$-stopping time and L_B is $\mathcal{F}_\infty$-measurable. Let

$$p_B(x) := P_x(\sigma_B < \infty).$$

Now, we can characterize recurrence and transience of $\mathbb{M}$ in terms of its sample paths behavior following [12]. More precisely, we have the following:

PROPOSITION **3.1** $\mathbb{M}$ *is transient, if and only if there exists a sequence of Borel finely open sets* $(B_n)_{n\geq 1}$ *increasing to* E *up to some* μ*-negligible set and for any* $n \geq 1$

$$P_x(L_{B_n} < \infty) = 1 \ \textit{for } \mu\textit{-a.e. } x \in E. \tag{3.6}$$

Proof For $g \in L^1(E,\mu)$ with $g \geq 0$ μ-a.e., define the potential operator of $(R_\alpha)_{\alpha>0}$ by

$$Rg(x) := E_x\left[\int_0^\infty g(X_t)dt\right], \ x \in E.$$

Assume that $\mathbb{M}$ is transient. Then, there exists $g \in L^1(E,\mu)$ with $g > 0$ μ-a.e. such that $Rg < \infty$ μ-a.e. Let

$$B_n := \left\{x \in E : Rg(x) > \frac{1}{n}\right\},\ n \geq 1.$$

Then, $(B_n)_{n\geq 1}$ are finely open and $(B_n)_{n\geq 1}$ increase to E up to some μ-negligible set. Let $(\theta_t)_{t\geq 0}$ be a shift operator of $\mathbb{M}$. Since $Rg < \infty$ μ-a.e., $p_t Rg \to 0$ μ-a.e. as $t \to \infty$ and $P_x(t + \sigma_{B_n} \circ \theta_t < \infty \text{ for any } t > 0) = 0$ μ-a.e., we have for $n \geq 1$ and $t > 0$

$$\begin{aligned} p_t Rg(x) &\geq p_{t+\sigma_{B_n}\circ\theta_t} Rg(x) \\ &= E_x\left[Rg(X_{t+\sigma_{B_n}\circ\theta_t})\ ;\ t + \sigma_{B_n} \circ \theta_t < \infty\right] \\ &= E_x\left[E_{X_t}\left[Rg(X_{\sigma_{B_n}})\right]\ ;\ \sigma_{B_n} < \infty\right] \\ &\geq \frac{1}{n} P_x\left(t + \sigma_{B_n} \circ \theta_t < \infty\right). \end{aligned}$$

Therefore, $P_x(L_{B_n} < \infty) = 1$ μ-a.e. $x \in E$.

Conversely, suppose there exists a sequence of Borel finely open sets $(B_n)_{n\geq 1}$ increasing to E up to some μ-negligible set satisfying (3.6). Let

$$g_{B_n}(x) := P_x(L_{B_n} > 0) = P_x(\sigma_{B_n} < \infty).$$

Then, we have $p_t g_{B_n}(x) = P_x(L_{B_n} > t) \nearrow g_{B_n}(x) \leq 1$ as $t \searrow 0$ which implies that $g_{B_n}(x)$ is excessive and bounded. By (3.6), we get for μ-a.e. $x \in E$,

$$p_t g_{B_n}(x) \to 0 \text{ as } t \to \infty.$$

Set

$$g_{nk}(x) := \frac{1}{k}\left(g_{B_n}(x) - p_k g_{B_n}(x)\right),$$

then we obtain $g_{nk}(x) \leq 1$ and for μ-a.e. $x \in E$,

$$Rg_{nk}(x) = \frac{1}{k} R(g_{B_n} - p_k g_{B_n})(x) = \frac{1}{k}\int_0^k p_t g_{B_n}(x)dt \leq 1.$$

Next, we will show that for μ-a.e. $x \in E$, there exist $n \geq 1$ and $k \geq 1$ such that $g_{nk}(x) > 0$. If $x \in B_n$, then $g_{B_n}(x) > 0$ because the sample paths are right continuous and B_n is a non-empty, finely open set. Fix $x \in B_n$. By (3.6),

$$(g_{B_n} - p_k g_{B_n})(x) = P_x(0 < L_{B_n} < k)$$

cannot be 0 for any $k \geq 1$. Let

$$\widetilde{g} := \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{g_{nk}}{2^n 2^k},$$

we get $\widetilde{g} > 0$ μ-a.e. and $R\widetilde{g} < \infty$ μ-a.e. Since μ is σ-finite, there exists $h \in L^1(E, \mu)$ with $h > 0$ μ-a.e. Then $g := \widetilde{g} \wedge h \in L^1(E, \mu)$ and $Rg < \infty$ μ-a.e. Therefore, $\mathbb{M}$ is transient.

□

A set B is called μ-polar, if

$$\int_E P_x(\sigma_B < \infty) \mu(dx) = 0.$$

PROPOSITION **3.2** *Let $\mathbb{M}$ be strictly irreducible and recurrent. Then we have the following properties:*

(i) Any bounded excessive function u satisfies for any $t > 0$,

$$p_t u(x) = u(x) \ \mu\text{-}a.e. \ x \in E.$$

(ii) Any excessive function is constant μ-a.e. on E.

(iii) $P_x(\zeta = \infty) = 1$ μ-a.e. $x \in E$.

(iv) If B is not μ-polar and finely open in E, then

$$P_x(L_B < \infty) = 0 \ \mu\text{-}a.e.\ x \in E.$$

Proof (i) Let u be a bounded and excessive function. Then $t \to p_t u$ is decreasing as $t \to \infty$. Let

$$\psi(x) := \lim_{t\to\infty} p_t u(x).$$

Then for any $s > 0$,

$$p_s\psi(x) = \lim_{t\to\infty} p_{s+t}u(x) = \psi(x).$$

Set $g := u - \psi$. Then for any $t > 0$,

$$p_t g = p_t u - p_t\psi = p_t u - \psi$$

and $p_t g \nearrow g$ as $t \searrow 0$, since u is excessive. It follows that g is also excessive and bounded. Furthermore, since $p_t g(x) \to 0$ as $t \to \infty$,

$$g_n := n(g - p_{1/n}g)$$

satisfies $Rg_n \nearrow g$ as $n \to \infty$. If $\mu(\{x \in E : g_n(x) > 0\}) > 0$, then $Rg_n = \infty$ μ-a.e. by the strict irreducibility and recurrence of $\mathbb{M}$. Since $g < \infty$, we can conclude $g_n = 0$ μ-a.e. for any $n \geq 1$. Thus $g = 0$ μ-a.e. Equivalently, $u = \psi$ μ-a.e. Since $u = \psi = \lim_{t\to\infty} p_t u \leq u$ for all $t > 0$,

$$p_t u(x) = u(x) \ \mu\text{-a.e. } x \in E.$$

(ii) Let B be a finely open set in E. Suppose $C := \{x \in E : p_B(x) = 1\}$ satisfies $\mu(C) > 0$ and set $D := \{x \in E : p_B(x) < 1\}$. Then D is finely open and $p_B = 1$ on C. Since p_B is bounded and excessive, for any $t > 0$,

$$1 = p_B(x) = p_t p_B(x) = E_x[p_B(X_t)] \quad \mu\text{-a.e. } x \in C.$$

Consequently, for each $t > 0$, $P_x(X_t \in D) = 0$ μ-a.e. $x \in C$. So we have

$$R1_D(x) = 0 \text{ for } \mu\text{-a.e. } x \in C.$$

If $\mu(D) > 0$, then by the strict irreducibility and recurrence of $\mathbb{M}$ we get $R1_D = \infty$ μ-a.e. Thus, $\mu(D) = 0$ and $p_B(x) = 1$ μ-a.e. $x \in E$. If there exists a non-constant excessive function f, then there exist constants $0 < a < b$ and finely open sets $A = \{x \in E : f(x) < a\}$ and $B = \{x \in E : f(x) > b\}$, such that

$$\mu(A) > 0 \text{ and } \mu(B) > 0.$$

Hence $p_B = 1$ μ-a.e. and for any $x \in A$, we have

$$\begin{aligned} a > f(x) \geq p_{\sigma_B} f(x) &= E_x[f(X_{\sigma_B})\ ;\ \sigma_B < \infty] \\ &\geq b\, P_x(\sigma_B < \infty) = b\, p_B(x). \end{aligned}$$

This is a contradiction to the fact that $\mu(A) > 0$ and $a < b$. Therefore, any excessive function is constant μ-a.e. on E.

(iii) Let $\psi(x) = E_x[1 - e^{-\zeta}]$. Then we have

$$p_t\psi(x) = E_x[\psi(X_t)\ ;\ t < \zeta] = E_x[1 - e^{-(\zeta - t)}\ ;\ t < \zeta] \ \nearrow\ \psi(x) \ as\ t \searrow 0.$$

Hence ψ is excessive and constant μ-a.e. on E by (ii). Therefore, $E_x[e^{-\zeta}] = c$ μ-a.e. $x \in E$ where c is a constant. Choose $x \in E$ and a non-empty open set $G \subset E$ with $x \notin \overline{G}$. Since G is finely open and not μ-polar, we get

$$p_G(x) = 1 \ \mu\text{-a.e. } x \in E.$$

Since $G \subset E$, we get $\{x \in E : p_G(x) < \infty\} = \{x \in E : p_G(x) < \zeta\}$. Hence,

$$\begin{aligned} c = E_x[e^{-\zeta}] &= E_x\left[e^{-\zeta}\ ;\ \sigma_G < \zeta\right] \\ &= E_x\left[e^{-\sigma_G} E_{X_{\sigma_G}}[e^{-\zeta}]\ ;\ \sigma_G < \zeta\right] \\ &= c\, E_x[e^{-\sigma_G}]. \end{aligned}$$

But $x \notin \overline{G}$ and so $E_x[e^{-\sigma_G}] < 1$. Therefore $E_x[e^{-\zeta}] = 0$ μ-a.e. $x \in E$.

(iv) Let B be not μ-polar and finely open. Then $p_B(x) = P_x(L_B > 0) = 1$ μ-a.e. $x \in E$. Let

$$\psi(x) = P_x(L_B < \infty).$$

Then $\psi(x) = P_x(0 < L_B < \infty)$ and $p_t\psi(x) = P_x(t < L_B < \infty)$ μ-a.e. $x \in E$. So ψ is excessive and bounded and satisfies $p_t\psi \to 0$ as $t \to \infty$. By (i) and (ii), there exists a constant c such that

$$\psi(x) = c \quad \text{and} \quad c = p_t c \quad \mu\text{-a.e. } x \in E.$$

Since $p_t\psi = p_t c \to 0$ as $t \to \infty$, we can conclude that $c = 0$, i.e. $P_x(L_B < \infty) = 0$ μ-a.e. $x \in E$.

□

Suppose that the process $\mathbb{M}$ is associated with $\mathcal{E}$, i.e. $R_\alpha f$ is a μ-version of $G_\alpha f$ for any $\alpha > 0$, $f \in B(E) \cap L^2(E, \mu)$. Then the strict irreducibility and recurrence of $(T_t)_{t>0}$ implies the strict irreducibility and recurrence of $\mathbb{M}$. Consequently, for any non-empty open set B,

$$P_x(\Lambda) = 1 \text{ for } \mu\text{-a.e. } x \in E,$$

where $\Lambda := \{\omega \in \Omega : L_B(\omega) = \infty\} \in \mathcal{F}_\infty$. Furthermore, assume that the semigroup p_t of $\mathbb{M}$ is *strong Feller* in the following sense: there exists a measurable function $(p_t(x, y))_{t>0, x, y \in E}$ with

$$E_x[f(X_t)] = p_t f(x) = \int_E p_t(x, y) f(y) \mu(dy)$$

for any $x \in E$, $f \in B(E)_b$ and

$$p_t f \text{ is continuous for any } f \in B(E)_b.$$

Since Λ is a shift invariant set, we can use the argument of [28, Lemma 7.1] to see that

$$P_x(\Lambda) = 1 \text{ for any } x \text{ in the support of } \mu.$$

Consequently, for an arbitrary non-empty open set B, the sample paths of $(X_t)_{t\geq 0}$ starting from any point x in the support of μ come back to B infinitely often.

Chapter 4 Applications on Euclidean space

Throughout this Chapter, we make the following assumptions:

Let $E \subset \mathbb{R}^d$ be either open or closed. If E is closed, we assume $dx(\partial E) = 0$ where E is the disjoint union of its interior E^0 and its boundary ∂E. Let $\varphi \in L^1_{loc}(E, dx)$ with $\varphi > 0$ dx-a.e. and $d\mu := \varphi dx$. Then μ is a σ-finite measure on $\mathcal{B}(E)$ and has full support. Let $C_0^\infty(E)$ be the set of infinitely often differentiable functions with compact support in E if E is open and $C_0^\infty(E) := \{u \in E \longrightarrow \mathbb{R} : \exists v \in C_0^\infty(\mathbb{R}^d) \text{ with } v = u \text{ on } E\}$ if E is closed. Let $\partial_i u$ denote the weak derivative of u with respect to x_i, $\nabla u := (\partial_1 u, \dots, \partial_d u)$, $|\cdot|$ the Euclidean norm and $\langle\ ,\ \rangle$ the Euclidean inner product.

Consider $A = (a_{ij})_{1\le i,j\le d} \in L^1_{loc}(E, \mu)$ with symmetric part $\widetilde{a}_{ij} := \frac{1}{2}(a_{ij} + a_{ji})$ and anti-symmetric part $\check{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji})$ and suppose that for each relatively compact open set $V \subset E$, i.e. V is relatively open in E and its closure $\overline{V}$ is compact and contained in E, there exists $\nu_V > 0$ such that

$$\nu_V^{-1}|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \nu_V |\xi|^2 \tag{4.1}$$

for all $\xi \in \mathbb{R}^d$, μ-a.e. $x \in V$. We assume further that

$$\mathcal{E}^0(f,g) := \int_E \langle A(x)\nabla f(x), \nabla g(x)\rangle \mu(dx),\ \ f, g \in C_0^\infty(E)$$

is closable on $L^2(E,\mu)$ and that $(\mathcal{E}^0, C_0^\infty(E))$ satisfies the strong sector condition, i.e. there is a constant $K > 0$ such that

$$\left|\mathcal{E}^0(f,g)\right| \le K\mathcal{E}^0(f,f)^{1/2}\mathcal{E}^0(g,g)^{1/2} \text{ for any } f, g \in C_0^\infty(E). \tag{4.2}$$

Denote the closure of $(\mathcal{E}^0, C_0^\infty(E))$ on $L^2(E,\mu)$ by $(\mathcal{E}^0, D(\mathcal{E}^0))$. Then $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a non-symmetric regular sectorial Dirichlet form on $L^2(E,\mu)$.

By $V \subset\subset E$, we mean that V is relatively compact open in E. For $V \subset\subset E$, let $C_0^\infty(V) := \{u \in C_0^\infty(E) : \mathrm{supp}(u) \subset V\}$. Since $(\mathcal{E}^0, C_0^\infty(E))$ is closable on $L^2(E,\mu)$, for any $V \subset\subset E$, $(\mathcal{E}^0, C_0^\infty(V))$ is also closable on $L^2(V,\mu)$. Denote its closure by $(\mathcal{E}^{0,V}, D(\mathcal{E}^{0,V}))$, then $D(\mathcal{E}^{0,V}) \subset D(\mathcal{E}^0)$ obviously. Furthermore by (4.1), the $\mathcal{E}_1^0$-norm is equivalent to the norm $\left(\int_V (u^2 + |\nabla u|^2) d\mu\right)^{1/2}$ on $D(\mathcal{E}^{0,V})$.

Let

$$D(\mathcal{E}^{0,E}) := \bigcup_{V \subset\subset E} D(\mathcal{E}^{0,V}),$$

i.e. $f \in D(\mathcal{E}^{0,E})$, if and only if there exists a subset $V \subset\subset E$ such that $f \in D(\mathcal{E}^{0,V})$. Note that $D(\mathcal{E}^{0,E}) \subset D(\mathcal{E}^0)$.

Let $(L^0, D(L^0))$ be the linear operator corresponding to $(\mathcal{E}^0, D(\mathcal{E}^0))$ on $L^2(E,\mu)$. By [18, I. Proposition 2.16], we know that $D(L^0) = \{u \in D(\mathcal{E}^0) : v \longmapsto \mathcal{E}^0(u,v)$ is continuous with respect to $\sqrt{(v,v)}$ on $D(\mathcal{E}^0)\}$ and that $\mathcal{E}^0(f,g) = (-L^0 f, g)$ for any $f \in D(L^0)$, $g \in D(\mathcal{E}^0)$. Let $(T_t^0)_{t>0}$ be the C_0-semigroup corresponding to $(L^0, D(L^0))$.

Let $B := (B_1, \dots, B_d) \in L^2_{loc}(E, \mathbb{R}^d, \mu)$ be μ-divergence free, i.e.

$$\int_E \langle B(x), \nabla f(x) \rangle \mu(dx) = 0 \tag{4.3}$$

for any $f \in C_0^\infty(E)$, hence for any $f \in D(\mathcal{E}^{0,E})$. Using the same technique as in [36], we can construct a closed extension $(\overline{L}, D(\overline{L}))$ of $Lu := L^0 u + \langle B, \nabla u \rangle$, $u \in D(L^0)_{0,b}$ on $L^1(E,\mu)$. For this, we need condition

(C) $D(L^0)_{0,b}$ is a dense subset of $L^1(E,\mu)$, which we assume from now on.

REMARK **4.1** *Condition (C) is needed to obtain strong continuity of the resolvent of* $(\overline{L}, D(\overline{L}))$*, exactly as it is obtained in [36] right after display (1.15). It is a weak condition. For instance, consider* $E := \mathbb{R}^d$ *and assume that the coefficients of the generator* L^0 *are locally square integrable with respect to the measure* μ *and that there are no boundary conditions. Then* $C_0^\infty(E) \subset D(L^0)_{0,b}$*, cf. e.g. Subsections 4.2.1, 4.2.2 and Remark 4.4 below. Condition (C) can even be obtained when the coefficients are not locally integrable with respect to the measure* μ *(see end of Remark 4.4). Similarly, one can obtain nice dense subsets of* D_0 *in case of boundary condition.*

LEMMA **4.1** *There exists a closed operator* $(\overline{L}, D(\overline{L}))$ *on* $L^1(E,\mu)$ *which is the generator of sub-Markovian* C_0*-semigroup of contractions* $(\overline{T}_t)_{t>0}$ *satisfying the following properties:*

(i) $(\overline{L}, D(\overline{L}))$ *is a closed extension of* $Lu = L^0u + \langle B, \nabla u\rangle$, $u \in D(L^0)_{0,b}$ *on* $L^1(E,\mu)$*.*

(ii) $D(\overline{L})_b \subset D(\mathcal{E}^0)$ *and for* $u \in D(\overline{L})_b$, $v \in D(\mathcal{E}^{0,E})_b$*, we have*

$$\mathcal{E}^0(u,v) - \int_E \langle B, \nabla u\rangle v d\mu = -\int_E \overline{L}uvd\mu$$

and

$$\mathcal{E}^0(u,u) \le -\int_E \overline{L}uud\mu.$$

Lemma 4.1 is proven in Chapter 5. Note that $(\overline{L}, D(\overline{L}))$ is a closed extension of $(L, D(L^0)_{0,b})$ on $L^1(E,\mu)$, but not necessarily the closure. Denote the C_0-resolvent of $(\overline{L}, D(\overline{L}))$ by $(\overline{G}_\alpha)_{\alpha>0}$. Since $(\overline{T}_t)_{t>0}$ is a sub-Markovian C_0-semigroup of contractions on $L^1(E,\mu)$ and $L^1(E,\mu)_b \subset L^2(E,\mu)$ densely, we can construct uniquely a sub-Markovian C_0-semigroup of contractions $(T_t)_{t>0}$ on $L^2(E,\mu)$ such that $T_t \equiv \overline{T}_t$ for $t>0$ on $L^1(E,\mu)\cap L^2(E,\mu)$ (cf. the Riesz-Thorin interpolation Theorem). Let $(L, D(L))$ be the generator of $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ be the corresponding C_0-resolvent. Clearly, $G_\alpha \equiv \overline{G}_\alpha$ on $L^1(E,\mu)\cap L^2(E,\mu)$ for $\alpha>0$. Let $(\widehat{L}, D(\widehat{L}))$ be the adjoint operator of $(L, D(L))$ in $L^2(E,\mu)$. Then

$$\mathcal{E}(f,g) := \begin{cases} (-Lf, g) & f \in D(L),\ g \in L^2(E,\mu), \\ (-\widehat{L}g, f) & g \in D(\widehat{L}),\ f \in L^2(E,\mu), \end{cases}$$

is a generalized Dirichlet form on $L^2(E,\mu)$ according to Chapter 2 with $\mathcal{A} \equiv 0$ on $\mathcal{V} = \mathcal{H} = L^2(E,\mu)$ and $(L, D(L)) = (\Lambda, D(\Lambda))$ (see also [37, I. Examples 4.9 (ii)]). Clearly (R1) holds since $(\widehat{L}, D(\widehat{L}))$ satisfies the same assumptions as $(L, D(L))$. In particular, the co-form

$$\widehat{\mathcal{E}}(f,g) := \mathcal{E}(g,f) \text{ for } (f,g) \in D(\widehat{L}) \times L^2(E,\mu) \cup L^2(E,\mu) \times D(L)$$

is also a generalized Dirichlet form. Though in general $\mathcal{E}$ is neither symmetric nor sectorial, it has the same fundamental properties as $\widehat{\mathcal{E}}$. Moreover, the bilinear form $\mathcal{E}$ is an extension of

$$\int_E \langle A\nabla f, \nabla g\rangle d\mu - \int_E \langle B, \nabla f\rangle g d\mu,$$

for $f, g \in \{f \in D(L^0)_{0,b} : \langle B, \nabla f\rangle \in L^2(E,\mu)\}$. Put

$$Nv := \langle B, \nabla v\rangle,\ v \in D(N) := D(\mathcal{E}^{0,E})_b.$$

Then $D = D(N) \cap D(\mathcal{E}^0) = D(N) = D(\mathcal{E}^{0,E})_b$ and $\mathcal{E}$ satisfies assumption (R2). Indeed, if $u \in D(L)_b$ and $v \in D$, then there exists a function $f \in L^2(E,\mu)$ such that $u = G_1 f$. We may assume that $f \geq 0$ μ-a.e. Otherwise, we put $u = u^+ - u^-$ where $u^+ := G_1 f^+$ and $u^- := G_1 f^-$. Choose an increasing sequence of functions $(f_n)_{n\geq 1} \subset L^1(E,\mu)_b$ such that $0 \leq f_n \nearrow f$ μ-a.e. as $n \to \infty$. Then $f_n \to f$ in $L^2(E,\mu)$ and $\overline{G}_1 f_n = G_1 f_n \to G_1 f$ in $L^2(E,\mu)$ as $n \to \infty$. Since $\overline{G}_1 f_n$ is increasing in n, we obtain $u_n := \overline{G}_1 f_n \leq G_1 f$ converges to $G_1 f$ μ-a.e. as $n \to \infty$. Thus, $(u_n)_{n\geq 1} \subset D(\overline{L})_b$ satisfies

$$u_n \to u \text{ in } L^2(E,\mu), \ \ \overline{L}u_n \to Lu \text{ in } L^2(E,\mu), \ \ u_n \nearrow u \ \mu\text{-a.e. as } n \to \infty$$

and $(u_n)_{n\geq 1}$ is uniformly bounded in n. Applying Lemma 4.1, we can see that

$$\sup_{n\geq 1} \mathcal{E}^0(u_n, u_n) < \infty$$

and so $u_n \to u$ weakly in $D(\mathcal{E}^0)$ as $n \to \infty$ as well as

$$\mathcal{E}^0(u,u) \leq \liminf_{n\to\infty} \mathcal{E}^0(u_n, u_n)$$

by [18, I. Lemma 2.12]. Hence using Lemma 4.1 and the approximation of u with $(u_n)_{n\geq 1}$, we obtain

$$\mathcal{E}^0(u,u) \leq \mathcal{E}(u,u), \ u \in D(L)_b$$

and

$$(-Lu, v) = \mathcal{E}^0(u,v) + \int_E \langle B, \nabla v\rangle u d\mu, \ u \in D(L)_b, \ v \in D \tag{4.4}$$

which achieves the proof that (R2) is satisfied. Consequently, by Theorem 3.1 and Corollary 3.1 of Section 3.1, we get the following facts.

COROLLARY **4.1**

(i) *If* $(\mathcal{E}^0, D(\mathcal{E}^0))$ *is transient, then* $(T_t)_{t>0}$ *is also transient.*

(ii) *If there exists a sequence of functions* $(\chi_n)_{n\geq 1} \subset D$ *with* $0 \leq \chi_n \leq 1$, $\lim_{n\to\infty} \chi_n = 1$ *μ-a.e. satisfying*

$$\lim_{n\to\infty} \left(\mathcal{E}^0(g, \chi_n) + \int_E \langle B, \nabla \chi_n \rangle g d\mu \right) = 0,$$

for any non-negative bounded g *in the extended Dirichlet space of* $D(\mathcal{E}^0)$, *then* $(T_t)_{t>0}$ *is not transient.*

REMARK **4.2** *If we construct a sequence of functions* $(\chi_n)_{n\geq 1} \subset D$ *with* $0 \leq \chi_n \leq 1$, $\lim_{n\to\infty} \chi_n = 1$ *μ-a.e. satisfying*

$$\lim_{n\to\infty} \left(\mathcal{E}^0(\chi_n, \chi_n) + \int_E |\langle B, \nabla \chi_n \rangle| d\mu \right) = 0,$$

then $(\chi_n)_{n\geq 1}$ *satisfies the conditions of Corollary 4.1(ii). Furthermore, since* $-B$ *satisfies the same assumptions as* B, *the co-form is then also not transient.*

Since $T_t \equiv \overline{T}_t$ for any $t > 0$ on $L^1(E, \mu) \cap L^2(E, \mu)$, it follows that the potential operator G obtained from $(T_t)_{t>0}$ (see paragraph right before Definition 3.1) is equal to the potential operator obtained from $(\overline{T}_t)_{t>0}$ (cf. Definition 3.1(iii)). Hence, the recurrence (resp. transience) of $(T_t)_{t>0}$ is equivalent to the recurrence (resp. transience) of $(\overline{T}_t)_{t>0}$. Next, we want to show that the recurrence of $(\overline{T}_t)_{t>0}$ implies the existence of a nice sequence of functions $(\chi_n)_{n\geq 1}$. This will be achieved in Theorem 4.1 below.

Let $h \in L^\infty(E,\mu)$ with $h \geq 0$ μ-a.e. and let $(\mathcal{E}^{0,h}, D(\mathcal{E}^0))$ be the bilinear form on $L^2(E,\mu)$ defined by

$$\mathcal{E}^{0,h}(f,g) := \mathcal{E}^0(f,g) + \int_E fghd\mu,\ f,g \in D(\mathcal{E}^0).$$

Since the $\mathcal{E}_1^{0,h}$- and $\mathcal{E}_1^0$-norms are equivalent on $D(\mathcal{E}^0)$, $(\mathcal{E}^{0,h}, D(\mathcal{E}^0))$ is also a regular Dirichlet form on $L^2(E,\mu)$. Let $(L^{0,h}, D(L^{0,h}))$ be the generator of $(\mathcal{E}^{0,h}, D(\mathcal{E}^0))$. Then $D(L^{0,h}) = D(L^0)$ and $L^{0,h}u = L^0u - h \cdot u$ for $u \in D(L^0) = D(L^{0,h})$. The following construction Lemma 4.2 is also proven in Chapter 5.

LEMMA **4.2** *There exists a closed operator $(\overline{L}^h, D(\overline{L}^h))$ on $L^1(E,\mu)$ which is the generator of sub-Markovian C_0-resolvent of contractions $(\overline{G}^h_\alpha)_{\alpha>0}$ satisfying the following properties:*

(i) $(\overline{L}^h, D(\overline{L}^h))$ *is a closed extension of* $L^h u := L^{0,h}u + \langle B, \nabla u\rangle$ $u \in D(L^0)_{0,b}$ *on* $L^1(E,\mu)$.

(ii) $D(\overline{L}^h)_b \subset D(\mathcal{E}^0)$ *and for* $u \in D(\overline{L}^h)_b$, $v \in D(\mathcal{E}^{0,E})_b$, *we have*

$$\mathcal{E}^{0,h}(u,v) - \int_E \langle B, \nabla u\rangle v d\mu = -\int_E \overline{L}^h uvd\mu$$

and

$$\mathcal{E}^{0,h}(u,u) \leq -\int_E \overline{L}^h uud\mu.$$

(iii) $D(\overline{L}^h) = D(\overline{L})$ *and for* $f \in L^1(E,\mu)$ *with* $f \geq 0$

$$\overline{G}^h_\alpha f = \overline{G}_\alpha(f - h\overline{G}^h_\alpha f).$$

Let $\varepsilon > 0$ be a constant and $h(\not\equiv \varepsilon)$ be as in the paragraph preceding Lemma 4.2. Consider the Hilbert space $L^2(E,(h+\varepsilon)\mu)$. Denote the inner product on $L^2(E,(h+\varepsilon)\mu)$ by $(\ ,\ )_{L^2(E,(h+\varepsilon)\mu)}$. Since

$$\varepsilon \cdot (f,f) \le (f,f)_{L^2(E,(h+\varepsilon)\mu)} \le (\varepsilon + \|h\|_{L^\infty(E,\mu)}) \cdot (f,f)$$

for any $f \in L^2(E,\mu)$, $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a regular Dirichlet form on $L^2(E,(h+\varepsilon)\mu)$ whose Dirichlet norm is equivalent to the norm of $(\mathcal{E}^0, D(\mathcal{E}^0))$ on $L^2(E,\mu)$. Let $(L^{0,\varepsilon}, D(L^{0,\varepsilon}))$ be the generator of $(\mathcal{E}^0, D(\mathcal{E}^0))$ on $L^2(E,(h+\varepsilon)\mu)$. Then $D(L^0) = D(L^{0,\varepsilon})$ and for $f \in D(L^0)$ and $g \in D(\mathcal{E}^0)$,

$$\mathcal{E}^0(f,g) = (-L^0 f, g) = (-L^{0,\varepsilon} f, g)_{L^2(E,(h+\varepsilon)\mu)}.$$

It follows that $L^{0,\varepsilon} f = \frac{1}{h+\varepsilon} L^0 f$ for any $f \in D(L^{0,\varepsilon})$. For $V \subset\subset E$, since $L^2(\mu)$- and $L^2((h+\varepsilon)\mu)$-norms are equivalent and $(\mathcal{E}^0, C_0^\infty(E))$ is closable on $L^2(E,\mu)$, $(\mathcal{E}^0, C_0^\infty(V))$ is also closable on $L^2(V,(h+\varepsilon)\mu)$. Denote the closure of $(\mathcal{E}^0, C_0^\infty(V))$ on $L^2(V,(h+\varepsilon)d\mu)$ by $(\mathcal{E}^{0,V}, D(\mathcal{E}^{0,\varepsilon,V}))$. Then, it is easy to show that $D(\mathcal{E}^{0,\varepsilon,V}) = D(\mathcal{E}^{0,V})$. Let

$$B^\varepsilon(x) := \frac{1}{h(x)+\varepsilon} B(x).$$

By (4.3),

$$\int_E \langle B^\varepsilon, \nabla f\rangle (h+\varepsilon) d\mu = 0$$

holds for any $f \in C_0^\infty(E)$.

LEMMA **4.3** *There exists a closed operator* $(\overline{L}^\varepsilon, D(\overline{L}^\varepsilon))$ *on* $L^1(E,(h+\varepsilon)\mu)$ *which is the generator of sub-Markovian* C_0*-resolvent of contractions* $(\overline{G}_\alpha^\varepsilon)_{\alpha>0}$ *satisfying the following properties:*

(i) $(\overline{L}^{\varepsilon}, D(\overline{L}^{\varepsilon}))$ *is a closed extension of* $L^{\varepsilon}u := L^{0,\varepsilon}u + \langle B^{\varepsilon}, \nabla u\rangle$ $u \in D(L^{0,\varepsilon})_{0,b}$ *on* $L^1(E, (h+\varepsilon)\mu)$.

(ii) $D(\overline{L}^{\varepsilon})_b \subset D(\mathcal{E}^0)$ *and for* $u \in D(\overline{L}^{\varepsilon})_b$, $v \in D(\mathcal{E}^{0,E})_b$, *we have*

$$\mathcal{E}^0(u,v) - \int_E \langle B^{\varepsilon}, \nabla u\rangle v(h+\varepsilon)d\mu = -\int_E \overline{L}^{\varepsilon} uv(h+\varepsilon)d\mu$$

and

$$\mathcal{E}^0(u,u) \le -\int_E \overline{L}^{\varepsilon} uu(h+\varepsilon)d\mu.$$

(iii) $D(\overline{L}^{\varepsilon}) = D(\overline{L})$ *and for* $f \in L^1(E, (h+\varepsilon)\mu)$ *with* $f \ge 0$ μ*-a.e.*

$$\overline{G}^{\varepsilon}_{\alpha} f = \overline{G}_{\alpha}((h+\varepsilon)f + \alpha(1-(h+\varepsilon))\overline{G}^{\varepsilon}_{\alpha} f).$$

Lemma 4.3 is also proven in Chapter 5. Lemmas 4.1, 4.2 and 4.3 assert that $D(\overline{L}) = D(\overline{L}^h) = D(\overline{L}^{\varepsilon})$ and for $u \in D(\overline{L})_b$, we have

$$\int_E (\overline{L}u - \overline{L}^h u - hu)v d\mu = 0 \quad \text{and} \quad \int_E (\overline{L}u - (h+\varepsilon)\overline{L}^{\varepsilon} u)v d\mu = 0,$$

for any $v \in D(\mathcal{E}^{0,E})_b$. Since $D(\mathcal{E}^{0,E})_b \subset L^{\infty}(E,\mu)$ densely, we obtain for any $u \in D(\overline{L})_b$,

$$\overline{L}^h u = \overline{L}u - h \cdot u \quad \text{and} \quad \overline{L}^{\varepsilon} u = \frac{1}{h+\varepsilon}\overline{L}u.$$

THEOREM **4.1** *If* $(\overline{T}_t)_{t>0}$ *is recurrent, then there exists a sequence of functions* $(\chi_n)_{n\ge 1} \subset D(\overline{L})_b$ *with* $0 \le \chi_n \le 1$ *such that*

$$\lim_{n\to\infty} \chi_n = 1 \ \mu\text{-a.e. and} \ \lim_{n\to\infty} \left(-\overline{L}\chi_n, \chi_n\right) = 0.$$

Furthermore, $\lim_{n\to\infty} -\overline{L}\chi_n = 0$ μ*-a.e. and in* $L^1(E,\mu)$*. In particular,* $(\widetilde{\mathcal{E}}^0, D(\mathcal{E}^0))$ *is recurrent (see [5, Theorem 1.6.3] and Lemma 4.1(ii)).*

Proof Let us choose $h \in L^1(E,\mu)_b$ with $h > 0$ μ-a.e. and $\varepsilon > 0$. Then we know that by Lemma 4.2, $\overline{G}^h_\varepsilon(\varepsilon f + fh) \in D(\overline{L})_b$ for any $f \in L^1(E,\mu)_b$ with $f \geq 0$. Observe

$$\begin{aligned}(1-\overline{L}^\varepsilon)\overline{G}^h_\varepsilon(\varepsilon f + fh) &= \overline{G}^h_\varepsilon(\varepsilon f + fh) - \frac{1}{h+\varepsilon}\overline{L}\,\overline{G}^h_\varepsilon(\varepsilon f + fh)\\ &= \overline{G}^h_\varepsilon(\varepsilon f + fh) - \frac{1}{h+\varepsilon}(\overline{L}^h - \varepsilon + h + \varepsilon)\overline{G}^h_\varepsilon(\varepsilon f + fh)\\ &= \overline{G}^h_\varepsilon(\varepsilon f + fh) + \frac{1}{h+\varepsilon}(\varepsilon f + fh) - \overline{G}^h_\varepsilon(\varepsilon f + fh) = f\end{aligned}$$

μ-a.e. Consequently, we obtain $\overline{G}^h_\varepsilon(\varepsilon f + fh) = \overline{G}^\varepsilon_1 f$. If $0 \leq f \leq 1$, then

$$0 \leq \overline{G}^h_\varepsilon(\varepsilon f + fh) \leq 1$$

for any $\varepsilon > 0$. Choosing $(f_n)_{n\geq 1} \subset L^1(E,\mu)_b$, $f_n \geq 0$ for $n \geq 1$, $f_n \nearrow 1$ μ-a.e. as $n \to \infty$, we obtain

$$0 \leq \overline{G}^h_\varepsilon h = \lim_{n\to\infty} \overline{G}^h_\varepsilon(f_n h) \leq \limsup_{n\to\infty} \overline{G}^h_\varepsilon(\varepsilon f_n + f_n h) \leq 1.$$

Let $\varepsilon \to 0$, then it follows $0 \leq G^h h \leq 1$ μ-a.e. where G^h is the potential operator associated with $(\overline{G}^h_\alpha)_{\alpha>0}$. Using Lemma 4.2(iii), we get

$$0 \leq G(h(1-G^h h)) = \lim_{\varepsilon\to 0}\overline{G}_\varepsilon(h(1-G^h h)) = \lim_{\varepsilon\to 0}\overline{G}_\varepsilon(h - h\overline{G}^h_\varepsilon h) = \lim_{\varepsilon\to 0}\overline{G}^h_\varepsilon h = G^h h \leq 1.$$

Since $(\overline{T}_t)_{t>0}$ is recurrent and $h(1-G^h h) \in L^1(E,\mu)_b$, hence by Definition 3.1(ii) $G(h(1-G^h h)) = 0$ μ-a.e. Consequently, $\overline{G}_1(h(1-G^h h)) = 0$ and by injectivity of $\overline{G}_1$, $G^h h = 1$ μ-a.e. Put $\chi_n := \overline{G}^h_{\frac{1}{n}} h$ for $n \geq 1$, then $0 \leq \chi_n \leq 1$ and $\chi_n \nearrow 1$ μ-a.e. as $n \to \infty$. Moreover for all $n \geq 1$,

$$\begin{aligned}0 \leq (-\overline{L}\chi_n, \chi_n) &= -\int_E \overline{L}\,\overline{G}^h_{\frac{1}{n}} h\chi_n d\mu = -\int_E \overline{L}\,\overline{G}_{\frac{1}{n}}(h - h\overline{G}^h_{\frac{1}{n}} h)\chi_n d\mu\\ &= -\frac{1}{n}\int_E \overline{G}_{\frac{1}{n}}(h - h\chi_n)\chi_n d\mu + \int_E (h - h\chi_n)\chi_n d\mu\\ &\leq \int_E h(1-\chi_n) d\mu\end{aligned}$$

and so $\lim_{n\to\infty}\left(-\overline{L}\chi_n,\chi_n\right)=0$.

□

As we have seen right before of Lemma 3.1, $(\overline{T}_t)_{t>0}$ is a sub-Markovian semigroup of contractions on $L^\infty(E,\mu)$.

DEFINITION **4.1** *$(\overline{T}_t)_{t>0}$ is said to be conservative if*

$$\overline{T}_t 1 = 1 \ \mu\text{-}a.e. \text{ for some (and hence any) } t>0.$$

It is well known that if the Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ is strictly irreducible recurrent, then it is conservative (cf. [5, Lemma 1.6.5] and [23, Corollary 1.3.8]). We have the following similar result in the non-sectorial situation of this Section.

COROLLARY **4.2** *If $(\overline{T}_t)_{t>0}$ is recurrent, then it is conservative.*

Proof Let $f\in L^1(E,\mu)_b$ with $f>0$. Then by Theorem 4.1, there exists $(\chi_n)_{n\geq1}\subset D(\overline{L})_b$ such that $\lim_{n\to\infty}-\overline{L}\chi_n=0$ in $L^1(E,\mu)$. Consequently, we obtain

$$\lim_{n\to\infty}\mathcal{E}(\chi_n,\widehat{G}_1 f)=\lim_{n\to\infty}-\int_E \overline{L}\chi_n\widehat{G}_1 f\,d\mu=0.$$

From this, the conservativeness of $(\overline{T}_t)_{t>0}$ follows by well-known standard arguments.

□

4.1 Explicit conditions for recurrence

Now, we shall find an explicit sequence of functions to determine recurrence of $(T_t)_{t>0}$. Assume that there exists a non-negative continuous function ρ on E with

$$\nabla\rho \in L^\infty_{loc}(E, \mathbb{R}^d, \mu)$$

such that for $r > 0$

$$E_r := \{x \in E : \rho(x) < r\}$$

is a relatively compact open set in E and $\bigcup_{r>0} E_r = E$. Then $\rho \in D(\mathcal{E}^0)_{loc}$. For instance, if E is closed and so in particular if $E = \mathbb{R}^d$, we may choose $\rho(x) = |x|$. Define for $r > 0$,

$$v_1(r) := \int_{E_r} \langle A(x)\nabla\rho(x), \nabla\rho(x)\rangle \mu(dx). \tag{4.5}$$

Since E_r is increasing in r, we may assume that $v_1(r) > 0$ for $r > 0$. From [33, Theorem 3], if

$$\int_1^\infty \frac{r}{v_1(r)} dr = \infty, \tag{4.6}$$

then the symmetric Dirichlet form $(\widetilde{\mathcal{E}}^0, D(\mathcal{E}^0))$ is recurrent. Furthermore, starting from (4.6) we can explicitly construct a sequence of functions $(\chi_n)_{n\geq 1} \subset D(\mathcal{E}^{0,E})_b$ such that $0 \leq \chi_n \leq 1$, $\lim_{n\to\infty} \chi_n = 1$ μ-a.e. and $\lim_{n\to\infty} \mathcal{E}^0(\chi_n, \chi_n) = 0$ which is an equivalent condition of recurrence of the symmetric Dirichlet form $(\widetilde{\mathcal{E}}^0, D(\mathcal{E}^0))$. Indeed, let

$$a_n := \int_1^n \frac{r}{v_1(r)} dr,$$

then $a_n \geq 0$, a_n is finite for all $n \geq 1$ and $\lim_{n\to\infty} a_n = \infty$. Let

$$\psi_n(r) := \begin{cases} 1 & 0 \leq r \leq 1, \\ 1 - \dfrac{1}{a_n}\displaystyle\int_1^r \frac{t}{v_1(t)}dt & 1 \leq r \leq n, \\ 0 & n \leq r. \end{cases}$$

Then $\lim_{n\to\infty} \psi_n(r) = 1$ dr-a.e. Let $\chi_n(x) := \psi_n(\rho(x))$. Since the support of $\psi_n(r)$ is $[0,n]$, the support of χ_n is $\overline{E}_n$. Similarly to [9, Theorem 2.2], we can show that $(\chi_n)_{n\geq 1} \subset D(\mathcal{E}^{0,E})_b$. We have $\nabla\chi_n(x) = -\frac{1}{a_n} 1_{\overline{E}_n \smallsetminus \overline{E}_1}(x) \frac{\rho(x)}{v_1(\rho(x))} \nabla\rho(x)$ for any $n \geq 1$. Hence by the transformation theorem for $n \geq 1$,

$$\mathcal{E}^0(\chi_n, \chi_n) = \int_{E_n \smallsetminus E_1} \langle A(x)\nabla\chi_n(x), \nabla\chi_n(x)\rangle \mu(dx) = \frac{1}{a_n^2}\int_1^n \frac{r^2}{v_1(r)^2}\nu_1(dr)$$

where ν_1 is the unique measure on $([0,\infty), \mathcal{B}([0,\infty)))$ which has v_1 as the distribution function. Let η be a standard mollifier on $\mathbb{R}$. Set $\eta_\varepsilon(r) = \frac{1}{\varepsilon}\eta\left(\frac{r}{\varepsilon}\right)$ so that $\int_{\mathbb{R}} \eta_\varepsilon(r)dr = 1$. Let

$$v_1^\varepsilon(r) := \int_{\mathbb{R}} v_1(r-t)\eta_\varepsilon(t)dt.$$

Since v_1 is continuous and strictly increasing, v_1^ε is also continuous and strictly increasing and v_1^ε uniformly converges to v_1 as $\varepsilon \to 0$ on each compact set in $[0,\infty)$. Let ν_1^ε be the unique measure on $([0,\infty), \mathcal{B}([0,\infty)))$ which has v_1^ε as the distribution function. Then, for any continuous function f and $n \geq 1$, we have

$$\int_1^n f(r)\nu_1^\varepsilon(dr) = \int_1^n f(r)v_1^\varepsilon(r)'dr \longrightarrow \int_1^n f(r)\nu_1(dr)$$

as $\varepsilon \to 0$. Consequently,

$$\begin{aligned}\int_1^n \frac{r^2}{v_1(r)^2}\nu_1(dr) &= \lim_{\varepsilon\to 0}\int_1^n \frac{r^2}{v_1(r)^2} v_1^\varepsilon(r)'dr\\ &= \lim_{\varepsilon\to 0}\int_1^n \frac{r^2}{v_1^\varepsilon(r)^2} v_1^\varepsilon(r)'dr\\ &= \lim_{\varepsilon\to 0}\int_1^n r^2\frac{d}{dr}\left(\frac{-1}{v_1^\varepsilon(r)}\right)dr\\ &= 2\int_1^n \frac{r}{v_1(r)}dr + \frac{1}{v_1(1)} - \frac{n^2}{v_1(n)}.\end{aligned}$$

Thus, we get

$$\mathcal{E}^0(\chi_n,\chi_n) \le \frac{2}{a_n} + \frac{1}{a_n^2 v_1(1)}.$$

Since the last term tends to 0 as $n \to \infty$, there exists a sequence of functions $(\chi_n)_{n\ge 1} \subset D(\mathcal{E}^{0,E})_b$ with $0 \le \chi_n \le 1$, $\lim_{n\to\infty}\chi_n = 1$ μ-a.e. satisfying

$$\lim_{n\to\infty}\mathcal{E}^0(\chi_n,\chi_n) = 0.$$

Now, we present an explicit sufficient condition for recurrence of $(T_t)_{t>0}$. Let

$$v_2(r) := \int_{E_r}\rho(x)\cdot|\langle B(x),\nabla\rho(x)\rangle|\mu(dx) \tag{4.7}$$

and ν_2 be the measure on $([0,\infty),\mathcal{B}([0,\infty)))$ which has v_2 as the distribution function. Let

$$v(r) := v_1(r) + v_2(r) \tag{4.8}$$

and ν be the measure on $([0,\infty),\mathcal{B}([0,\infty)))$ which has v as the distribution function. Then it is easy to see that $\nu(A) \ge \nu_i(A)$ for $A \in \mathcal{B}([0,\infty))$, $i = 1,2$.

THEOREM **4.2** *Let v_1, v_2 and v be defined as in (4.5), (4.7) and (4.8) respectively. If the sequence $(a_n)_{n\ge 1}$ defined by*

$$a_n := \int_1^n \frac{r}{v(r)}dr$$

satisfies

$$\lim_{n\to\infty} a_n = \infty \ \textit{and} \ \lim_{n\to\infty} \frac{\log(v_2(n)\vee 1)}{a_n} = 0,$$

then $(T_t)_{t>0}$ *is not transient. In particular, if* $(T_t)_{t>0}$ *is additionally strictly irreducible, then* $(T_t)_{t>0}$ *is recurrent.*

Proof In view of Corollary 3.1(ii), the last assertion follows from the first one. Concerning the first one, it follows from Remark 4.2, that it suffices to construct a sequence of functions $(\chi_n)_{n\geq 1}\subset D(\mathcal{E}^{0,E})_b$ with $0\leq \chi_n\leq 1$, $\lim_{n\to\infty}\chi_n = 1$ μ-a.e. satisfying

$$\lim_{n\to\infty}\left(\mathcal{E}^0(\chi_n,\chi_n) + \int_E |\langle B,\nabla\chi_n\rangle| d\mu\right) = 0. \tag{4.9}$$

First, assume that $B\not\equiv 0$ μ-a.e. For $r>0$, let

$$\psi_n(r) := \begin{cases} 1 & 0\leq r\leq 1,\\ 1-\dfrac{1}{a_n}\displaystyle\int_1^r \frac{t}{v(t)}dt & 1\leq r\leq n,\\ 0 & n\leq r.\end{cases}$$

Then $\lim_{n\to\infty}\psi_n(r) = 1$ dr-a.e. Let $\chi_n(x) := \psi_n(\rho(x))$. Then $(\chi_n)_{n\geq 1}\subset D(\mathcal{E}^{0,E})_b$. We have $\nabla\chi_n(x) = -\frac{1}{a_n}1_{\overline{E}_n\setminus\overline{E}_1}(x)\frac{\rho(x)}{v(\rho(x))}\nabla\rho(x)$ for any $n\geq 1$. Hence for $n\geq 1$,

$$\begin{aligned}\mathcal{E}^0(\chi_n,\chi_n) + \int_E |\langle B,\nabla\chi_n\rangle| d\mu &= \int_{E_n\setminus E_1} \langle A(x)\nabla\chi_n(x),\nabla\chi_n(x)\rangle + |\langle B(x),\nabla\chi_n(x)\rangle|\mu(dx)\\ &= \frac{1}{a_n^2}\int_{E_n\setminus E_1}\frac{\rho(x)^2}{v(\rho(x))^2}\langle A(x)\nabla\rho(x),\nabla\rho(x)\rangle\mu(dx)\\ &\quad + \frac{1}{a_n}\int_{E_n\setminus E_1}\frac{\rho(x)}{v(\rho(x))}|\langle B(x),\nabla\rho(x)\rangle|\mu(dx)\\ &= \frac{1}{a_n^2}\int_1^n \frac{r^2}{v(r)^2}\nu_1(dr) + \frac{1}{a_n}\int_1^n\frac{1}{v(r)}\nu_2(dr)\\ &\leq \frac{1}{a_n^2}\int_1^n \frac{r^2}{v(r)^2}\nu(dr) + \frac{1}{a_n}\int_1^n\frac{1}{v_2(r)}\nu_2(dr)\\ &\leq \frac{2}{a_n} + \frac{1}{a_n^2 v(1)} + \frac{\log(v_2(n)\vee 1)}{a_n}.\end{aligned}$$

By the assumptions on $(a_n)_{n\geq 1}$, the last term tends to 0 as $n \to \infty$. Consequently, $(T_t)_{t>0}$ is recurrent. If $B \equiv 0$ μ-a.e., then $\log(v_2(n) \vee 1) \equiv 0$ and (4.9) also holds.

$\square$

COROLLARY **4.3** *Let v_1, v_2, and v be defined as in (4.5), (4.7), and (4.8) respectively. The conditions on $(a_n)_{n\geq 1}$ in Theorem 4.2 are satisfied, if one of the following conditions is fulfilled for sufficiently large r:*

(i) $v_1(r) \leq br^2$ *and* $v_2(r) \leq b \log r$ *for some constant* $b > 0$

(ii) $v(r) \leq cr^\alpha$ *for some constants* $c > 0$ *and* $\alpha < 2$.

Consequently, if either (i) or (ii) holds, then $(T_t)_{t>0}$ is not transient. In particular, if $(T_t)_{t>0}$ is additionally strictly irreducible, then $(T_t)_{t>0}$ is recurrent.

4.2 Examples and counterexamples

In this Section, we provide explicit examples and counterexamples. We start with several counterexamples which show that the existence of $(\chi_n)_{n\geq 1} \subset D(\mathcal{E}^0)$ such that $0 \leq \chi_n \leq 1$, $\lim_{n\to\infty} \chi_n = 1$ μ-a.e. and

$$\lim_{n\to\infty} \mathcal{E}(\chi_n, \chi_n) = 0$$

is not a sufficient condition for recurrence of $(T_t)_{t>0}$ in contrast to the symmetric case where this is always true (cf. [5, Theorem 1.6.3]). At the end of this Section, we discuss recurrence and transience related to Muckenhoupt weights.

4.2.1 A counterexample using results from [36]

Consider the case where $E = \mathbb{R}$ and $(\mathcal{E}^0, D(\mathcal{E}^0))$ is given as the closure of

$$\mathcal{E}^0(f,g) := \int_{\mathbb{R}} f'(x)g'(x)\mu(dx),\ \ f,g \in C_0^\infty(\mathbb{R})$$

on $L^2(\mathbb{R},\mu)$ where $d\mu := e^{-x^2}dx$. Then it is easy to see that $1 \in D(L^0)$, $L^0 1 = 0$ and $C_0^\infty(\mathbb{R}) \subset D(L^0)$. In particular, condition (C) is satisfied. Moreover, $B(x) := -6e^{x^2}$ satisfies (4.3) and so by Lemma 4.1, we can construct a closed operator $(\overline{L}, D(\overline{L}))$ which is a closed extension of $Lu := L^0 u + Bu'$, $u \in D(L^0)_{0,b}$ on $L^1(\mathbb{R},\mu)$ satisfying (i) and (ii) in Lemma 4.1. By [36, Remark 1.11 and Example 1.12], $(\overline{T}_t)_{t>0}$ is not conservative, hence not recurrent by Corollary 4.2. Since $\mu(\mathbb{R}) < \infty$, the restriction of $(\overline{T}_t)_{t>0}$ on $L^2(\mathbb{R},\mu)$ coincides with the $L^2(\mathbb{R},\mu)$-semigroup $(T_t)_{t>0}$. Thus, $(L, D(L))$ is given as the part of $(\overline{L}, D(\overline{L}))$ on $L^2(\mathbb{R},\mu)$, i.e.

$$D(L) = \{u \in L^2(\mathbb{R},\mu) \cap D(\overline{L}) : \overline{L}u \in L^2(\mathbb{R},\mu)\}$$

and

$$Lf := \overline{L}f \quad f \in D(L).$$

Let $D := D(\mathcal{E}^0)_{0,b}$. Then for $f \in D(L)_b$, $g \in D$, we have by (4.4),

$$\mathcal{E}(f,g) = (-Lf, g) = \mathcal{E}^0(f,g) + \int_{\mathbb{R}} Bg' f d\mu.$$

and

$$\mathcal{E}^0(f,f) \le \mathcal{E}(f,f)$$

where g' denotes the derivative of g. Thus, $\mathcal{E}$ satisfies (R1) and (R2). By construction of $(\overline{L}, D(\overline{L}))$, we have

$$D(L^0)_{0,b} \subset D(\overline{L})_{0,b}$$

and if $u \in D(L^0)_{0,b}$, then $u \in L^2(\mathbb{R}, \mu)$ and

$$\overline{L}u = L^0 u + Bu' \in L^2(\mathbb{R}, \mu).$$

Consequently, $C_0^\infty(\mathbb{R}) \subset D(L^0)_{0,b} \subset D(L)_{0,b}$. Choose $(\chi_n)_{n\geq 1} \subset C_0^\infty(\mathbb{R})$ such that $0 \leq \chi_n \leq 1$, $\lim_{n\to\infty} \chi_n = 1$ μ-a.e. and $\|\chi_n'\|_{L^\infty(\mathbb{R},\mu)} \leq 2/n$. It then follows from (4.3), that

$$\lim_{n\to\infty} (-L\chi_n, \chi_n) = \lim_{n\to\infty} \mathcal{E}(\chi_n, \chi_n) = \lim_{n\to\infty} \mathcal{E}^0(\chi_n, \chi_n) = 0.$$

4.2.2 A generic counterexample

We call the following counterexample generic, since it works for a large class of φ. We let hence $E = \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}^+$ be locally bounded above and below by strictly positive constants with $\varphi' \in L^2_{loc}(\mathbb{R}, dx)$, $d\mu = \varphi dx$ and $B(x) = \frac{b}{\varphi(x)}$ for some constant $b \neq 0$. Note that these general assumptions on φ imply that $C_0^\infty(\mathbb{R}) \subset D(L^0)_{0,b} \subset D(L)_{0,b}$ and that

$$Lf = \frac{1}{2}f'' + \left(\frac{\varphi'}{2\varphi} + B\right) f', \ \ f \in C_0^\infty(\mathbb{R}).$$

These two facts are important for our arguments below. In particular, condition (C) is satisfied. Using similar arguments as in Subsection 4.2.1, we can construct a generalized Dirichlet form $\mathcal{E}$ satisfying (R1) and (R2) and such that $\mathcal{E}$ is given as an extension of

$$\mathcal{E}(f,g) := \frac{1}{2}\int_{\mathbb{R}} f'(x)g'(x)\mu(dx) - \int_{\mathbb{R}} B(x)f'(x)g(x)\mu(dx), \ \ f,g \in C_0^\infty(\mathbb{R})$$

on $L^2(\mathbb{R}, \mu)$. By the specialties of dimension one, $\mathcal{E}$ can be symmetrized, i.e. there exists a symmetric Dirichlet form $(\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}}))$ whose infinitesimal generator $(\widetilde{L}, D(\widetilde{L}))$ coincides with $(L, D(L))$ locally. This will be realized in (4.13) below.

For $n \geq 1$, let $V_n := (-n, n)$ be the open interval from $-n$ to n in $\mathbb{R}$ and $(\mathcal{E}^{0,n}, D(\mathcal{E}^{0,n}))$ be the symmetric Dirichlet form given as the closure of

$$\mathcal{E}^{0,n}(f,g) = \frac{1}{2}\int_{V_n} f'g'd\mu, \ \ f,g \in C_0^\infty(V_n).$$

Let $(L^{0,n}, D(L^{0,n}))$ be the closed linear operator on $L^2(V_n,\mu)$ corresponding to $(\mathcal{E}^{0,n}, D(\mathcal{E}^{0,n}))$. Since $C_0^\infty(\mathbb{R}) \subset D(L^0)_{0,b}$ and B satisfies (4.3), by [36, Proposition 1.1], we can construct a closed operator $(\overline{L}^n, D(\overline{L}^n))$ which is the closure of $L^n u = L^{0,n}u + Bu'$, $u \in D(L^{0,n})_b$ on $L^1(V_n,\mu)$. Let $(L^n, D(L^n))$ be the part of $(\overline{L}^n, D(\overline{L}^n))$ on $L^2(V_n,\mu)$, then we have $D(L^{0,n})_b \subset D(L^n)$,

$$L^n f = \overline{L}^n f = L^{0,n}f + Bf', \ \ f \in D(L^{0,n})_b \tag{4.10}$$

and

$$\mathcal{E}^{0,n}(f,g) - \int_{V_n} Bf'gd\mu = -\int_{V_n} L^n fgd\mu, \ \ f \in D(L^n)_b, \ \ g \in D(\mathcal{E}^{0,n}).$$

Let $(G_\alpha^n)_{\alpha>0}$ be the C_0-resolvent of contractions corresponding to $(L^n, D(L^n))$ on $L^2(V_n,\mu)$. Since the $L^2(\mu)$- and $L^2(dx)$-norms are equivalent on V_n, $D(\mathcal{E}^{0,n}) = H_0^{1,2}(V_n, dx) :=$ the closure of $C_0^\infty(V_n)$ with respect to the norm $\left(\int_{V_n}(u^2+(u')^2)dx\right)^{1/2}$ in $L^2(V_n,dx)$. Thus, $u \in D(\mathcal{E}^{0,n})$, if and only if u is equal a.e. to an absolutely continuous function which has a.e. an ordinary derivative belonging to $L^2(V_n,dx)$ and u does not have a boundary value, i.e. for any $v \in C_0^\infty(\overline{V}_n)$,

$$\int_{V_n} u'vdx = -\int_{V_n} uv'dx.$$

Let

$$\widetilde{\varphi}(x) := \exp\left(\int_0^x \frac{\varphi'(s)+2b}{\varphi(s)}ds\right) \tag{4.11}$$

and $(\widetilde{\mathcal{E}}^n, D(\mathcal{E}^{0,n}))$ be the bilinear form on $L^2(V_n, \widetilde{\varphi}dx)$ defined by

$$\widetilde{\mathcal{E}}^n(f,g) := \frac{1}{2}\int_{V_n} f'g'\widetilde{\varphi}dx, \;\; f,g \in D(\mathcal{E}^{0,n}).$$

Since the $L^2(\mu)$- and $L^2(\widetilde{\varphi}dx)$-norms are equivalent on V_n, $(\widetilde{\mathcal{E}}^n, D(\mathcal{E}^{0,n}))$ is a symmetric Dirichlet form on $L^2(V_n, \widetilde{\varphi}dx)$. Let $(\widetilde{L}^{V_n}, D(\widetilde{L}^{V_n}))$ be the closed linear operator on $L^2(V_n, \widetilde{\varphi}dx)$ corresponding to $(\widetilde{\mathcal{E}}^n, D(\mathcal{E}^{0,n}))$.

LEMMA **4.4** *$D(\widetilde{L}^n) = D(L^{0,n})$ and for $u \in D(\widetilde{L}^n)$,*

$$\widetilde{L}^n u = L^{0,n}u + Bu'.$$

Proof Suppose that $u \in D(L^{0,n})$. We first show that $u \in D(\widetilde{L}^n)$, i.e. $v \longmapsto \widetilde{\mathcal{E}}^n(u,v)$ is continuous with respect to $\sqrt{(v,v)_{L^2(V_n,\widetilde{\varphi}dx)}}$ on $D(\mathcal{E}^{0,n})$. Let

$$\psi(x) := \exp\left(\int_0^x \frac{2b}{\varphi(s)}ds\right).$$

It is easy to see that $\widetilde{\varphi} = \psi\varphi$ and that $v\psi \in D(\mathcal{E}^{0,n})$ for any $v \in D(\mathcal{E}^{0,n})$. Then

$$\mathcal{E}^{0,n}(u, v\psi) = \frac{1}{2}\int_{V_n} u'(v\psi)'\varphi dx = \frac{1}{2}\int_{V_n} u'v'\psi\varphi dx + \frac{1}{2}\int_{V_n} u'\psi' v\varphi dx.$$

Consequently, we obtain

$$\widetilde{\mathcal{E}}^n(u,v) = -\int_{V_n} L^{0,n}u \cdot v\psi\varphi dx - \int_{V_n} \frac{b}{\varphi}u'v\psi\varphi dx,$$

hence $u \in D(\widetilde{L}^n)$ and

$$\widetilde{L}^n u = L^{0,n}u + Bu'.$$

Conversely, suppose that $u \in D(\widetilde{L}^n)$ and that $v \in D(\mathcal{E}^{0,n})$. Then $\frac{v}{\psi} \in D(\mathcal{E}^{0,n})$ and

$$\widetilde{\mathcal{E}}^n\left(u, \frac{v}{\psi}\right) = \frac{1}{2}\int_{V_n} u'\left(\frac{v}{\psi}\right)'\widetilde{\varphi}dx = \frac{1}{2}\int_{V_n} u'v'\varphi dx - \frac{1}{2}\int_{V_n} u'v\frac{\psi'}{\psi^2}\widetilde{\varphi}dx.$$

Consequently, we obtain

$$\mathcal{E}^{0,n}(u,v) = -\int_{V_n} \widetilde{L}^n u \cdot \frac{v}{\psi}\widetilde{\varphi}dx + \int_{V_n} \frac{b}{\varphi}u'\frac{v}{\psi}\widetilde{\varphi}dx$$

and so $u \in D(L^{0,n})$.

$\square$

Let $(\widetilde{G}^n_\alpha)_{\alpha>0}$ be the C_0-resolvent of contractions corresponding to $(\widetilde{L}^n, D(\widetilde{L}^n))$. By Lemma 4.4 and (4.10), we obtain $D(\widetilde{L}^n) = D(L^{0,n})$ and for any $u \in D(L^{0,n})_b$,

$$L^n u = \widetilde{L}^n u.$$

Since $(\widetilde{L}^n, D(\widetilde{L}^n))$ is a Dirichlet operator on $L^2(V_n, \widetilde{\varphi}dx)$, we get $D(\widetilde{L}^n)_b \subset D(\widetilde{L}^n)$ densely and so by Lemma 4.4,

$$(1 - L^n)D(L^{0,n})_b = (1 - \widetilde{L}^n)D(\widetilde{L}^n)_b \subset L^2(V_n, \widetilde{\varphi}dx) = L^2(V_n, \mu)$$

densely. Consequently, we obtain $D(\widetilde{L}^n) = D(L^n)$ and for $u \in D(L^n)$,

$$L^n u = \widetilde{L}^n u.$$

It follows that for $f \in L^2(V_n, \mu)$,

$$G^n_\alpha f = (\alpha - L^n)^{-1} f = (\alpha - \widetilde{L}^n)^{-1} f = \widetilde{G}^n_\alpha f, \ \mu\text{-a.e.} \tag{4.12}$$

The C_0-resolvent of contractions $(\overline{G}_\alpha)_{\alpha>0}$ of $(\overline{L}, D(\overline{L}))$ is defined by

$$\overline{G}_\alpha f = \lim_{n\to\infty} \overline{G}^n_\alpha (f \cdot 1_{V_n}), \ \mu\text{-a.e. } f \in L^1(\mathbb{R}, \mu)$$

(cf. proof of Lemma 4.1 in Chapter 5). For the C_0-resolvent of contractions $(G_\alpha)_{\alpha>0}$ of $(L, D(L))$, it holds (see right after Lemma 4.1)

$$G_\alpha f = \overline{G}_\alpha f = \lim_{n\to\infty} \overline{G}^n_\alpha (f \cdot 1_{V_n}) = \lim_{n\to\infty} G^n_\alpha (f \cdot 1_{V_n}), \ \mu\text{-a.e. } f \in L^1(\mathbb{R}, \mu) \cap L^2(\mathbb{R}, \mu),$$

hence, $G_\alpha f = \lim_{n\to\infty} G^n_\alpha(f\cdot 1_{V_n}),\ f\in L^2(\mathbb{R},\mu)$.

Next, we will construct a symmetric Dirichlet form on $L^2(\mathbb{R},\widetilde{\varphi}dx)$ which extends $(\widetilde{\mathcal{E}}^n, D(\mathcal{E}^{0,n}))$ for any $n\geq 1$. We have already constructed a sub-Markovian C_0-resolvent of contractions $(\widetilde{G}^n_\alpha)_{\alpha>0}$ on $L^2(V_n,\widetilde{\varphi}dx)$. For $f\in L^2(\mathbb{R},\widetilde{\varphi}dx)$, $\alpha>0$

$$\widetilde{G}_\alpha f := \lim_{n\to\infty} \widetilde{G}^n_\alpha(f\cdot 1_{V_n})$$

exists $\widetilde{\varphi}dx$-a.e. and $(\widetilde{G}_\alpha)_{\alpha>0}$ is a sub-Markovian C_0-resolvent of contractions on $L^2(\mathbb{R},\widetilde{\varphi}dx)$ (cf. proof of Lemma 4.1). Since for each $n\geq 1$, $(\widetilde{G}^n_\alpha)_{\alpha>0}$ is symmetric, so is $(\widetilde{G}_\alpha)_{\alpha>0}$. Let $(\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}}))$ be the symmetric Dirichlet form corresponding to $(\widetilde{G}_\alpha)_{\alpha>0}$. Then $(\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}}))$ is a closed extension of

$$\frac{1}{2}\int_{\mathbb{R}} f'g'\widetilde{\varphi}dx,\ \ f,g\in C_0^\infty(\mathbb{R}).$$

For $f\in L^2(\mathbb{R},\mu)\cap L^2(\mathbb{R},\widetilde{\varphi}dx)$, using the above and (4.12) it holds

$$G_\alpha f = \lim_{n\to\infty} G^n_\alpha(f\cdot 1_{V_n}) = \lim_{n\to\infty} \widetilde{G}^n_\alpha(f\cdot 1_{V_n}) = \widetilde{G}_\alpha f,\ \ \mu\text{-a.e.} \tag{4.13}$$

Therefore, the potential operators of $(G_\alpha)_{\alpha>0}$ and $(\widetilde{G}_\alpha)_{\alpha>0}$ are the same on $L^1(\mathbb{R},\mu)\cap L^1(\mathbb{R},\widetilde{\varphi}dx)$ and the recurrence or transience of $(G_\alpha)_{\alpha>0}$ and $(\widetilde{G}_\alpha)_{\alpha>0}$ are equivalent.

REMARK **4.3** *If we choose φ and b so that either it holds*

$$\int_0^\infty \frac{1}{\widetilde{\varphi}(x)}dx<\infty \quad \textit{or} \quad \int_{-\infty}^0 \frac{1}{\widetilde{\varphi}(x)}dx<\infty, \tag{4.14}$$

where $\widetilde{\varphi}$ is as in (4.11), then it follows similarly to [26, Theorem 3.11] that $\widetilde{\mathcal{E}}$ is not recurrent. Consequently, $\mathcal{E}$ is also not recurrent. However, as in Subsection 4.2.1, there exists a sequence of functions $(\chi_n)_{n\geq 1}\subset C_0^\infty(\mathbb{R})$, such that $0\leq\chi_n\leq 1$, $\lim_{n\to\infty}\chi_n = 1$ μ-a.e. and

$$\lim_{n\to\infty}\mathcal{E}(\chi_n,\chi_n)=0.$$

For instance, if $\varphi(x) = e^{-|x|}$, $b = \frac{1}{2}$, then

$$\widetilde{\varphi}(x) = e^{-x}\exp((e^{x}-1)) \text{ for } x \geq 0.$$

Consequently,

$$\int_0^\infty \frac{1}{\widetilde{\varphi}(x)}dx = \int_0^\infty \frac{e^x}{\exp(e^x-1)}dx = 1,$$

and so the criterion (4.14) of Remark 4.3 is satisfied. Moreover, it is easy to see that for this choice of φ and b, $\mathcal{E}$ has the following additional properties: $\mathcal{E}$ is not conservative and $\mathcal{E}$ does not satisfy the weak sector condition, i.e. it holds

$$\sup_{u,v\in C_0^\infty(\mathbb{R})\setminus\{0\}} \frac{|(-Lu,v)|}{\|u\|_{D(\mathcal{E}^0)}\|v\|_{D(\mathcal{E}^0)}} = \infty.$$

Replacing $\varphi(x) = e^{-|x|}$ by $\varphi(x) = \min\left\{1, \frac{1}{|x|}\right\}$ the criterion (4.14) of Remark 4.3 is still satisfied, but $\mathcal{E}$ becomes conservative and does not satisfy the strong sector condition, i.e. it holds

$$\sup_{u,v\in C_0^\infty(\mathbb{R})\setminus\{0\}} \frac{|(-Lu,v)|}{\sqrt{\mathcal{E}^0(u,u)}\sqrt{\mathcal{E}^0(v,v)}} = \infty.$$

However, in this case, it is not easy to see whether $\mathcal{E}$ satisfies the weak sector condition or not.

EXAMPLE **4.1** *Choosing $\varphi(x) \equiv 1$ and $B(x) \equiv b$ for some constant $b \neq 0$, gives another example where the criterion (4.14) of Remark 4.3 is satisfied. Hence, $\mathcal{E}$ is not recurrent, but there exists a sequence of functions $(\chi_n)_{n\geq 1} \subset C_0^\infty(\mathbb{R})$ such that $0 \leq \chi_n \leq 1$, $\lim_{n\to\infty}\chi_n = 1$ μ-a.e. and*

$$\lim_{n\to\infty} \mathcal{E}(\chi_n,\chi_n) = 0. \tag{4.15}$$

However, by [36, Proposition 1.10(c)], dx is $(\overline{T}_t)$-invariant. This example shows that even though (4.15) holds and the reference measure dx is $(\overline{T}_t)$-invariant, $(T_t)_{t>0}$ does not need to be recurrent. Obviously, in this example $\mathcal{E}$ satisfies the weak sector condition, but not the strong sector condition, i.e. $\mathcal{E}$ is not sectorial in the sense of this thesis.

4.2.3 Muckenhoupt weights

In this Subsection, we present a class of examples of φ and B applying Corollary 4.1(i) and Corollary 4.3. Consider the case where $E = \mathbb{R}^d$ with $d \geq 2$ and $(\mathcal{E}^0, D(\mathcal{E}^0))$ is given as the closure of

$$\mathcal{E}^0(f,g) := \int_{\mathbb{R}^d} \langle \nabla f, \nabla g \rangle d\mu, \;\; f,g \in C_0^\infty(\mathbb{R}^d)$$

on $L^2(\mathbb{R}^d, \mu)$, where $d\mu := \varphi dx$ and φ is an $\mathcal{A}_\beta$-weight, $\beta \in [1,2]$ (cf. [43, Definition 1.2.2]). Note that for $\varphi \in \mathcal{A}_\beta$ (short for φ is an $\mathcal{A}_\beta$-weight), $\beta \in [1,2]$, the closability follows since $\mathcal{A}_\beta \subset \mathcal{A}_2$ and $\frac{1}{\varphi} \in L^1_{loc}(\mathbb{R}^d, dx)$ for any $\varphi \in \mathcal{A}_2$ (cf. [43, Remark 1.2.4]). Assume that $B \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d, \mu)$ satisfies (4.3) and that $D(L^0)_{0,b}$ is a dense subset of $L^1(\mathbb{R}^d, \mu)$, i.e. condition (C) is satisfied.

REMARK **4.4** *For instance, if $\varphi = \xi^2$, $\xi \in H^{1,2}_{loc}(\mathbb{R}^d, dx)$, $\varphi > 0$ dx-a.e. where $H^{1,2}(\mathbb{R}^d, dx)$ denote the usual Sobolev space of order one in $L^2(\mathbb{R}^d, dx)$ and $H^{1,2}_{loc}(\mathbb{R}^d, dx) := \{f : f \cdot \chi \in H^{1,2}(\mathbb{R}^d, dx)$ for any $\chi \in C_0^\infty(\mathbb{R}^d)\}$, then $C_0^\infty(\mathbb{R}^d) \subset D(L^0)$ and (C) holds. Another example is given in 4.2.3(c) below, where the drift coefficient may even not be in $L^1_{loc}(\mathbb{R}^d, \mu)$, i.e. in the non-semimartingale case.*

Under the present assumptions, we can construct a generalized Dirichlet form $\mathcal{E}$ satisfying (R1) and (R2) and which is an extension of

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^d} \langle \nabla f(x), \nabla g(x) \rangle \mu(dx) - \int_{\mathbb{R}^d} \langle B(x), \nabla f(x) \rangle g(x) \mu(dx)$$

for $f, g \in \{f \in D(L^0)_{0,b} : \langle B, \nabla f \rangle \in L^2(\mathbb{R}^d, \mu)\}$. We consider the following condition on B:

There exist constants $M > 0$ and $\alpha \in \mathbb{R}$ such that

$$|\langle B(x), x \rangle| \leq M(1 + |x|)^\alpha$$

for μ-a.e. sufficiently large $|x|$.

(a) Let φ be a Muckenhoupt $\mathcal{A}_1$-weight and $d = 2$. By [43, Proposition 1.2.7], for $r > 1$ and some constant A

$$v_1(r) \leq A r^2.$$

Since $\varphi \in \mathcal{A}_1$, there exists $p > 1$ such that $\varphi^p \in \mathcal{A}_1$ (cf. [41, IX. Theorem 3.5 (Reverse Hölder)]). We may assume that $\alpha \cdot \frac{p}{p-1} + 2 \neq 0$ (otherwise choose a slightly bigger p). Note that for sufficiently large $r > 1$,

$$\begin{aligned} v_2(r) = \int_{B_r} |\langle B(x), x \rangle| \varphi(x) dx &\leq \left(\int_{B_r} \varphi^p dx \right)^{\frac{1}{p}} \left(\int_{B_r} |\langle B(x), x \rangle|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &\leq C(1+r)^{\alpha + \frac{2}{p} + \frac{2p-2}{p}} = C(1+r)^{\alpha+2}, \end{aligned}$$

where C is some positive constant. By Corollary 4.3, if $\alpha \leq -2$, then $\mathcal{E}$ is not transient.

(b) Let φ be a Muckenhoupt $\mathcal{A}_\beta$-weight with $1 \leq \beta \leq 2$. Then the assumptions (A), (B) and (C) in [34] are satisfied on $\mathbb{R}^d$ for $(\mathcal{E}^0, D(\mathcal{E}^0))$ (cf. [29, Lemma 5.2]). Furthermore, by [34, Proposition 2.3] and [35, Section 4], there exists a measurable function $(p_t^0(x,y))_{t>0,x,y\in\mathbb{R}^d}$ and some constant $C>0$ depending on β, d and A such that

$$T_t^0 f(x) = \int_{\mathbb{R}^d} p_t^0(x,y) f(y)\mu(dy) \ \ \mu\text{-a.e. } x \in \mathbb{R}^d,$$

where $f \in L^2(\mathbb{R}^d,\mu)$ and $(T_t^0)_{t>0}$ denotes the C_0-semigroup of contractions on $L^2(\mathbb{R}^d,\mu)$ corresponding to $(\mathcal{E}^0, D(\mathcal{E}^0))$ and for any $t>0$ $x,y\in\mathbb{R}^d$,

$$\frac{1}{C\cdot\mu(B_{\sqrt{t}}(y))}\exp\left(-\frac{C|x-y|^2}{t}\right) \le p_t^0(x,y) \le \frac{C\left(1+\frac{|x-y|^2}{t}\right)^{\frac{\beta d+\log_2 A}{2}}}{\sqrt{\mu(B_{\sqrt{t}}(x))\mu(B_{\sqrt{t}}(y))}}\exp\left(-\frac{|x-y|^2}{4t}\right).$$

By Remark 3.1(iii), $(\mathcal{E}^0, D(\mathcal{E}^0))$ is irreducible. Consequently, by [35, Corollary 4.12], Remark 3.1(ii) and (iv)

$$\int_1^\infty \frac{r}{v_1(r)}dr < \infty, \tag{4.16}$$

if and only if $(\mathcal{E}^0, D(\mathcal{E}^0))$ is transient. Hence, by Corollary 4.1(i), (4.16) is a sufficient criterion for transience of $\mathcal{E}$.

(c) Let $\varphi(x) := |x|^\eta$ with $-d<\eta$. Note that then $C_0^\infty(\mathbb{R}^d\setminus\{0\}) \subset D(L^0)_{0,b}$ for any $\eta>-d$, hence (C) holds. Then

$$v_1(r) = \int_{B_r}|x|^\eta dx = C_1 r^{d+\eta},$$

where C_1 depends on d and for sufficiently large $r>1$,

$$v_2(r) = \int_{B_r}|\langle B(x),x\rangle|\cdot|x|^\eta dx \le \begin{cases} C_2(1+r)^{\alpha+d+\eta} & \alpha+d+\eta\neq 0,\\ C_2\log(1+r) & \alpha+d+\eta=0,\end{cases}$$

where C_2 depends on d and M. For $-d<\eta<d$, it is well-known that $\varphi\in\mathcal{A}_2$. By (b) (cf. (4.16)), if $-d+2<\eta<d$, then $\mathcal{E}$ is transient. Moreover by Corollary 4.3, if one of the following conditions is satisfied, then $\mathcal{E}$ is not transient.

(c1) $d + \eta = 2$ and $\alpha \leq -2$.

(c2) $d + \eta \in (0, 2)$ and $\alpha + d + \eta < 2$.

Similarly to [36, Section 3] one can show that there exists a diffusion process associated with $\mathcal{E}$ and similarly to [42, Theorem 4.5] one can then derive a semimartingale characterization of this process. In particular, if $d+\eta \in (0, 1]$, then the associated process will not be semimartingale. Thus (c2) asserts that we are able to determine non-transience or recurrence of this process even in the non semimartingale case.

4.3 Explicit recurrence criteria for symmetric Dirichlet forms on $\mathbb{R}$ satisfying a Hamza type condition

In this Section, we present sufficient conditions for recurrence of symmetric Dirichlet forms which are strongly local and hence associated to diffusions on $\mathbb{R}$ with reflecting boundary conditions and without boundary conditions (see [5, Theorem 1.6.3]). Our main achievement is that the explicit results are obtained under quite weak assumptions on the closability, hence regularity of the underlying coefficients.

4.3.1 Non-reflected case

Let μ be the σ-finite measure defined by $d\mu = \varphi dx$ with $\varphi \in L^1_{loc}(\mathbb{R}, dx)$ with $\varphi > 0$ dx-a.e. Let σ be a measurable function such that $\sigma > 0$ dx-a.e. and such that $\sigma\varphi \in L^1_{loc}(\mathbb{R}, dx)$. Consider the symmetric bilinear form

$$\mathcal{E}(f, g) := \frac{1}{2} \int_{\mathbb{R}} \sigma(x) f'(x) g'(x) \mu(dx), \quad f, g \in C_0^\infty(\mathbb{R})$$

on $L^2(\mathbb{R},\mu)$. Let U be the largest open set in $\mathbb{R}$ such that

$$\frac{1}{\sigma\varphi} \in L^1_{loc}(U, dx).$$

Assume

$$dx(\mathbb{R} \setminus U) = 0. \tag{4.17}$$

Furthermore, we suppose $(\mathcal{E}, C_0^\infty(\mathbb{R}))$ is closable on $L^2(\mathbb{R}, d\mu)$. For instance, if there exists some open set $\tilde{U} \subset \mathbb{R}$ such that

$$\frac{1}{\varphi} \in L^1_{loc}(\tilde{U}, dx) \quad \text{and} \quad dx(\mathbb{R} \setminus (U \cap \tilde{U})) = 0,$$

then $(\mathcal{E}, C_0^\infty(\mathbb{R}))$ is closable on $L^2(\mathbb{R},\mu)$ by [18, II, 2 a)]. Denote the closure by $(\mathcal{E},\mathcal{F})$. We present sufficient conditions for the recurrence of the symmetric Dirichlet form $(\mathcal{E},\mathcal{F})$.

REMARK **4.5** *Under quite weak regularity assumptions on σ and φ one can show using integration by parts that the generator $(L, D(L))$ corresponding to $(\mathcal{E},\mathcal{F})$ (for the definition of generator see [5] or [18]) is given by*

$$Lf = \frac{\sigma}{2} f'' + \frac{1}{2}\left(\sigma' + \sigma\frac{\varphi'}{\varphi}\right) f',$$

i.e. for $f \in D(L) \subset \mathcal{F}$ we have

$$-\int_{\mathbb{R}} Lfg\,d\mu = \mathcal{E}(f,g).$$

In particular, (assuming again that everything is sufficiently regular) choosing

$$\varphi(x) = \frac{1}{\sigma(x)} \exp\left(\int_0^x \frac{2b}{\sigma}(s)ds\right)$$

for some (freely chosen) function b, we get

$$Lf = \frac{\sigma}{2} f'' + bf'.$$

Therefore, our framework is suitable for the description of diffusion type operators in one dimension with concrete coefficients.

Since $U \subset \mathbb{R}$ is open, U is the disjoint union (we use the symbol $\dot\cup$ to denote this) of countably (finite or infinite) many open intervals. There are five possible cases that we summarize in the following theorem.

THEOREM **4.3** *$(\mathcal{E}, \mathcal{F})$ is recurrent if one of the following conditions holds:*

(i) $U = (-\infty, \infty)$ *and*

$$\int_{-\infty}^{0} \frac{1}{\sigma\varphi}(s)ds = \int_{0}^{\infty} \frac{1}{\sigma\varphi}(s)ds = \infty.$$

(ii) $U = (-\infty, a) \dot\cup V \dot\cup (b, \infty)$ *where* V *is some open set (so either* V *is empty if* $a = b$*, or* V *is non-empty if* $a < b$*) and*

$$\int_{-\infty}^{a-1} \frac{1}{\sigma\varphi}(s)ds = \int_{b+1}^{\infty} \frac{1}{\sigma\varphi}(s)ds = \infty.$$

(iii) $U = \bigcup_{n\geq 1} I_{-n} \dot\cup V \dot\cup \bigcup_{n\geq 1} I_n$*, where* $I_n = (x_n, x_{n+1})$*,* $x_n < x_{n+1}$*,* $I_{-n} = (x_{-n}, x_{-n+1})$*,* $x_{-n} < x_{-n+1}$*,* $n \geq 1$*,* V *is some open set and*

$$a_n := \int_{c_n}^{d_n} \frac{1}{\sigma\varphi}(s)ds \longrightarrow \infty$$
$$a_{-n} = b_n := \int_{d_{-n}}^{c_{-n}} \frac{1}{\sigma\varphi}(s)ds \longrightarrow \infty$$

as $n \to \infty$*, where*

$$c_n = \frac{x_{n+1} + x_n}{2}, \quad n \in \mathbb{Z} \setminus \{0\},$$

and for $n \geq 1$

$$d_n := \begin{cases} x_{n+1} - \frac{1}{n} & x_{n+1} - c_n > 1, \\ x_{n+1} - \frac{x_{n+1} - c_n}{n} & x_{n+1} - c_n \leq 1, \end{cases}$$

$$d_{-n} := \begin{cases} x_{-n} + \frac{1}{n} & c_{-n} - x_{-n} > 1, \\ x_{-n} + \frac{c_{-n} - x_{-n}}{n} & c_{-n} - x_{-n} \leq 1. \end{cases}$$

(iv) $U = \bigcup_{n \geq 1} I_{-n} \,\dot\cup\, V \,\dot\cup\, (b, \infty)$, *where* $I_{-n} = (x_{-n}, x_{-n+1})$, $x_{-n} < x_{-n+1}$, $n \geq 1$, V *is some open set,*

$$\int_{b+1}^{\infty} \frac{1}{\sigma\varphi}(s)ds = \infty,$$

and

$$a_{-n} = b_n := \int_{d_{-n}}^{c_{-n}} \frac{1}{\sigma\varphi}(s)ds \longrightarrow \infty$$

as $n \to \infty$ *where for* $n \geq 1$

$$c_{-n} = \frac{x_{-n} + x_{-n+1}}{2}$$

and

$$d_{-n}(x) := \begin{cases} x_{-n} + \frac{1}{n} & c_{-n} - x_{-n} > 1, \\ x_{-n} + \frac{c_{-n} - x_{-n}}{n} & c_{-n} - x_{-n} \leq 1. \end{cases}$$

(v) $U = (-\infty, a) \,\dot\cup\, V \,\dot\cup\, \bigcup_{n \geq 1} I_n$, *where* $I_n = (x_n, x_{n+1})$, $x_n < x_{n+1}$, $n \geq 1$, V *is some open set,*

$$\int_{-\infty}^{a-1} \frac{1}{\sigma\varphi}(s)ds = \infty,$$

and

$$a_n := \int_{c_n}^{d_n} \frac{1}{\sigma\varphi}(s)ds \longrightarrow \infty$$

as $n \to \infty$ *where for* $n \geq 1$

$$c_n = \frac{x_{n+1} + x_n}{2}$$

and

$$d_n := \begin{cases} x_{n+1} - \frac{1}{n} & x_{n+1} - c_n > 1, \\ x_{n+1} - \frac{x_{n+1} - c_n}{n} & x_{n+1} - c_n \leq 1. \end{cases}$$

Proof (i) By [5, Theorem 1.6.3], it suffices to find a sequence $(\chi_n)_{n\geq 1} \subset \mathcal{F}$ with $0 \leq \chi_n \leq 1$, $\chi_n \nearrow 1$ as $n \to \infty$ and $\lim_{n\to\infty} \mathcal{E}(\chi_n, \chi_n) = 0$. Define sequences a_n and b_n by

$$a_n := \int_0^n \frac{1}{\sigma\varphi}(s)ds \quad \text{and} \quad b_n := \int_{-n}^0 \frac{1}{\sigma\varphi}(s)ds.$$

Since $1/\sigma\varphi \in L^1_{loc}(\mathbb{R}, dx)$, a_n and b_n are well defined and converge to ∞. Let

$$\chi_n(x) := \begin{cases} 1 - \frac{1}{a_n}\int_0^x \frac{1}{\sigma\varphi}(t)dt & x \in [0, n], \\ 1 - \frac{1}{b_n}\int_x^0 \frac{1}{\sigma\varphi}(t)dt & x \in [-n, 0], \\ 0 & \text{elsewhere.} \end{cases}$$

For each $n \geq 1$, χ_n has compact support and is bounded. Moreover, since $1/\sigma\varphi \in L^1_{loc}(\mathbb{R}, dx)$, $\chi_n(x)$ is differentiable at every Lebesgue point of $1/\sigma\varphi$, hence dx-a.e. Furthermore, it is easy to see that $0 \leq \chi_n \nearrow 1$ dx-a.e. as $n \to \infty$ and $\chi_n'(x)$ exists dx-a.e. with

$$\chi_n'(x) = \begin{cases} -\frac{1}{a_n}\frac{1}{\sigma\varphi}(x) & x \in [0, n], \\ \frac{1}{b_n}\frac{1}{\sigma\varphi}(x) & x \in [-n, 0], \\ 0 & \text{elsewhere.} \end{cases}$$

Now, it remains to show that $(\chi_n)_{n\geq 1} \subset \mathcal{F}$. Let η be a standard mollifier on $\mathbb{R}$. Set $\eta_\varepsilon(x) = \frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right)$ so that $\int_{\mathbb{R}} \eta_\varepsilon dx = 1$ and so that the support of η_ε is in

$(-\varepsilon, \varepsilon)$. Then $\eta_\varepsilon * \chi_n \in C_0^\infty(\mathbb{R})$ and $(\eta_\varepsilon * \chi_n)' = \eta_\varepsilon * \chi_n'$, $n \geq 1$. We have

$$(\eta_\varepsilon * \chi_n(x) - \chi_n(x)) = \int_{\mathbb{R}} [\chi_n(x-y) - \chi_n(x)]\eta_\varepsilon(y)dy,$$

$$|\eta_\varepsilon * \chi_n(x) - \chi_n(x)|^2 \leq \int_{\mathbb{R}} |\chi_n(x-y) - \chi_n(x)|^2 \eta_\varepsilon(y)dy,$$

and

$$\begin{aligned}\int_{\mathbb{R}} (\eta_\varepsilon * \chi_n(x) - \chi_n(x))^2 \varphi(x)dx &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\chi_n(x-y) - \chi_n(x)|^2 \eta_\varepsilon(y)dy\varphi(x)dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} |\chi_n(x-y) - \chi_n(x)|^2 \varphi(x)dx\eta_\varepsilon(y)dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} |\chi_n(x-\varepsilon y) - \chi_n(x)|^2 \varphi(x)dx\eta(y)dy.\end{aligned}$$

Let

$$g(y) := \int_{\mathbb{R}} |\chi_n(x-y) - \chi_n(x)|^2 \varphi(x)dx,$$

then $g(0) = 0$ and g is continuous and bounded. Thus by Lebesgue, $\lim_{\varepsilon\to 0} \eta_\varepsilon * \chi_n = \chi_n$ in $L^2(\mathbb{R}, \mu)$. Furthermore, for $0 < \varepsilon < 1$

$$\begin{aligned}\mathcal{E}(\eta_\varepsilon * \chi_n, \eta_\varepsilon * \chi_n) &= \int_{\mathbb{R}} \left(\eta_\varepsilon * \chi_n'(x)\right)^2 \sigma(x)\varphi(x)dx \\ &\leq \|\chi_n'\|^2_{L^1(\mathbb{R},dx)} \int_{-n-1}^{n+1} \sigma(x)\varphi(x)dx\end{aligned}$$

Since $|\eta_\varepsilon * \chi_n'(x)| \leq \|\chi_n'\|_{L^1(\mathbb{R},dx)} < \infty$, we have

$$\sup_{0<\varepsilon<1} \mathcal{E}(\eta_\varepsilon * \chi_n, \eta_\varepsilon * \chi_n) < \infty.$$

Thus, from [18, Lemma 2.12], $(\chi_n)_{n\geq 1} \subset \mathcal{F}$ and we get

$$\begin{aligned}\mathcal{E}(\chi_n, \chi_n) &= \frac{1}{2} \int_{\mathbb{R}} \chi_n'(x)\chi_n'(x)\sigma(x)\varphi(x)dx \\ &= \frac{1}{2}\left[\frac{1}{a_n^2}\int_0^n \frac{1}{\sigma\varphi}(x)dx + \frac{1}{b_n^2}\int_{-n}^0 \frac{1}{\sigma\varphi}(x)dx\right] \\ &= \frac{1}{2}\left[\frac{1}{a_n} + \frac{1}{b_n}\right].\end{aligned}$$

Therefore, $\lim_{n\to\infty}\mathcal{E}(\chi_n,\chi_n)=0$. i.e. $(\mathcal{E},\mathcal{F})$ is recurrent.

(ii) For $n\geq 1$, let

$$\chi_n(x) := \begin{cases} 1 & x\in[a-1,b+1], \\ 1-\frac{1}{a_n}\int_{b+1}^{x}\frac{1}{\sigma\varphi}(t)dt & x\in[b+1,b+1+n], \\ 1-\frac{1}{b_n}\int_{x}^{a-1}\frac{1}{\sigma\varphi}(t)dt & x\in[a-1-n,a-1], \\ 0 & \text{elsewhere}, \end{cases}$$

where

$$a_n=\int_{b+1}^{b+1+n}\frac{1}{\sigma\varphi}(s)ds \quad\text{and}\quad b_n=\int_{a-1-n}^{a-1}\frac{1}{\sigma\varphi}(s)ds.$$

Then $(\chi_n)_{n\geq 1}\subset\mathcal{F}$ satisfies the desired properties and determines recurrence.

(iii) Let

$$\chi_n(x) := \begin{cases} 1 & x\in[c_{-n},c_n], \\ 1-\frac{1}{a_n}\int_{c_n}^{x}\frac{1}{\sigma\varphi}(t)dt & x\in[c_n,d_n], \\ 1-\frac{1}{b_n}\int_{x}^{c_{-n}}\frac{1}{\sigma\varphi}(t)dt & x\in[d_{-n},c_{-n}], \\ 0 & \text{elsewhere}, \end{cases}$$

where

$$a_n=\int_{c_n}^{d_n}\frac{1}{\sigma\varphi}(s)ds \quad\text{and}\quad b_n=\int_{d_{-n}}^{c_{-n}}\frac{1}{\sigma\varphi}(s)ds.$$

Then $(\chi_n)_{n\geq 1}\subset\mathcal{F}$ satisfies the desired properties and determines recurrence.

(iv) and (v) are combinations of (ii) and (iii) and are proved by combining the proofs of (ii) and (iii).

□

REMARK **4.6** *With the obvious modifications Theorem 4.3 can be reformulated for Dirichlet forms that are given as the closure of*

$$\frac{1}{2}\int_V \sigma(x)f'(x)g'(x)\mu(dx), \quad f,g \in C_0^\infty(V)$$

where V is an arbitrary open and connected set in $\mathbb{R}$. We omit this here to avoid trivial complications.

REMARK **4.7** *Note that we do not assume that $(\mathcal{E},\mathcal{F})$ is irreducible. As a non-trivial example consider the following: Let $S = \{x_i \in \mathbb{R} : i \in \mathbb{Z}\}$ with $x_i < x_{i+1}$ for all $i \in \mathbb{Z}$ and assume S does not have an accumulation point in $\mathbb{R}$. For $\alpha \geq 1$, define a function φ by*

$$\varphi(x) = |x - x_i|^\alpha, \ x \in \left[\frac{x_i + x_{i-1}}{2}, \frac{x_i + x_{i+1}}{2}\right], \ i \in \mathbb{Z}.$$

Then $\varphi > 0$ on $\mathbb{R} \setminus S$, hence dx-a.e. Assume $\sigma \equiv 1$, then since $1/\varphi \in L^1_{loc}(\mathbb{R} \setminus S, dx)$, (4.17) is also satisfied. Thus, the symmetric bilinear form $(\mathcal{E}, C_0^\infty(\mathbb{R}))$ defined by

$$\mathcal{E}(f,g) \coloneqq \frac{1}{2}\int_{\mathbb{R}} f'(x)g'(x)\mu(dx), \quad f,g \in C_0^\infty(\mathbb{R})$$

is closable on $L^2(\mathbb{R}, d\mu)$ where $d\mu = \varphi dx$. Define the sequences a_n, b_n, c_n and d_n as in the Theorem 4.3(iii), then

$$a_n = \int_{c_n}^{d_n} \frac{1}{\varphi}(s)ds = \int_{c_n}^{d_n} \frac{1}{(x_{n+1} - s)^\alpha} ds.$$

(i) If $\alpha = 1$, then

$$a_n = -\log(x_{n+1} - s)\big|_{c_n}^{d_n} = -\log(x_{n+1} - d_n) + \log(x_{n+1} - c_n).$$

In this case, if $x_{n+1} - c_n > 1$, then $x_{n+1} - d_n = \frac{1}{n}$ and

$$a_n = -\log\left(\frac{1}{n}\right) + \log(x_{n+1} - c_n) > \log n.$$

If $x_{n+1} - c_n \le 1$, *then* $x_{n+1} - d_n = \frac{x_{n+1}-c_n}{n}$ *and*

$$a_n = -\log\left(\frac{x_{n+1}-c_n}{n}\right) + \log(x_{n+1}-c_n) = \log n.$$

Thus, we get $\lim_{n\to\infty} a_n = \infty$.

(ii) If $\alpha > 1$, *then*

$$\begin{aligned} a_n &= \int_{c_n}^{d_n} \frac{1}{(x_{n+1}-s)^\alpha} ds \\ &= \frac{-1}{1-\alpha}(x_{n+1}-s)^{1-\alpha}|_{c_n}^{d_n} \\ &= \frac{1}{\alpha-1}\left[(x_{n+1}-d_n)^{1-\alpha} - (x_{n+1}-c_n)^{1-\alpha}\right]. \end{aligned}$$

In this case, if $x_{n+1} - c_n > 1$, *then* $x_{n+1} - d_n = \frac{1}{n}$ *and*

$$a_n = \frac{1}{\alpha-1}\left[\left(\frac{1}{n}\right)^{1-\alpha} - (x_{n+1}-c_n)^{1-\alpha}\right] > \frac{1}{\alpha-1}(n^{\alpha-1}-1).$$

If $x_{n+1} - c_n \le 1$, *then* $x_{n+1} - d_n = \frac{x_{n+1}-c_n}{n}$ *and*

$$\begin{aligned} a_n &= \frac{1}{\alpha-1}\left[\left(\frac{x_{n+1}-c_n}{n}\right)^{1-\alpha} - (x_{n+1}-c_n)^{1-\alpha}\right] \\ &= \frac{1}{\alpha-1}\left[(x_{n+1}-c_n)^{1-\alpha}\left\{\left(\frac{1}{n}\right)^{1-\alpha} - 1\right\}\right] \\ &\ge \frac{1}{\alpha-1}(n^{\alpha-1}-1). \end{aligned}$$

Thus, $\lim_{n\to\infty} a_n = \infty$ *if* $\alpha \ge 1$. *In the same way,* $\lim_{n\to\infty} b_n = \infty$. *Therefore,* $(\mathcal{E},\mathcal{F})$ *is recurrent, thus in particular conservative (cf. [5, Theorems 1.6.5 and 1.6.6]). Since* $(\mathcal{E},\mathcal{F})$ *is also strongly local, the process associated to* $(\mathcal{E},\mathcal{F})$ *is a conservative diffusion (cf. [5]). Moreover, since by [5, Example 3.3.2]* $Cap(\{x_i\}) = 0$, *the sets* (x_i, x_{i+1}) *are all invariant for any* $i \in \mathbb{Z}$, *i.e.*

$$p_t 1_{(x_i,x_{i+1})}(x) = 0, \quad x \notin (x_i, x_{i+1}), \text{ for any } i \in \mathbb{Z}$$

where p_t is the transition semigroup of (the process associated to) the Dirichlet form $(\mathcal{E},\mathcal{F})$. But

$$\mu((x_i,x_{i+1})) \neq 0 \quad \textit{and} \quad \mu(\mathbb{R}\setminus(x_i,x_{i+1})) \neq 0 \quad \textit{for any } i\in\mathbb{Z}.$$

Therefore, $(\mathcal{E},\mathcal{F})$ is not irreducible (in the sense of [5]).

4.3.2 Reflected case

Let $I = [0,\infty)$ and $C_0^\infty(I) := \{f : I \to \mathbb{R} : \exists g \in C_0^\infty(\mathbb{R}) \text{ with } g = f \text{ on } I\}$. Let $\varphi \in L^1_{loc}(I,dx)$ with $\varphi > 0$ dx-a.e. Furthermore, assume σ is a measurable function such that $\sigma > 0$ dx-a.e. and such that $\sigma\varphi \in L^1_{loc}(I,dx)$. Consider the symmetric bilinear form

$$\mathcal{E}(f,g) := \frac{1}{2}\int_I \sigma(x)f'(x)g'(x)\mu(dx), \quad f,g\in C_0^\infty(I)$$

on $L^2(I,\mu)$ where $\mu := \varphi dx$. As in the Section 4.3, let U be the largest open set in I such that

$$\frac{1}{\sigma\varphi} \in L^1_{loc}(U,dx)$$

and assume

$$dx(I\setminus U) = 0. \tag{4.18}$$

We suppose $(\mathcal{E},C_0^\infty(I))$ is closable on $L^2(I,\mu)$. For instance, if there is some open set $\tilde{U}\subset I$ such that $1/\varphi \in L^1_{loc}(\tilde{U},dx)$ and $dx(I\setminus(U\cap\tilde{U})) = 0$, then $(\mathcal{E},C_0^\infty(I))$ is closable on $L^2(I,\mu)$ by the results of [42, Lemma 1.1]. Denote the closure by $(\mathcal{E},\mathcal{F})$.

There are two possible cases that we summarize in the following theorem.

THEOREM **4.4** $(\mathcal{E}, D(\mathcal{E}))$ *is recurrent if one of the following conditions holds:*

(i) $U = V \dot{\cup} (a, \infty)$ *where* V *is some open set (so either* V *is empty if* $a = 0$*, or* V *is non-empty if* $a > 0$*) and*

$$\int_{a+1}^{\infty} \frac{1}{\sigma\varphi}(s)ds = \infty.$$

(ii) $U = V \dot{\cup} \bigcup_{n\geq 1} I_n$*, where* $I_n = (x_n, x_{n+1})$*,* $x_n < x_{n+1}$*,* $n \geq 1$*,* V *is some open set and*

$$a_n := \int_{c_n}^{d_n} \frac{1}{\sigma\varphi}(s)ds \longrightarrow \infty$$

as $n \to \infty$*, where for* $n \geq 1$

$$c_n = \frac{x_{n+1} + x_n}{2}$$

and

$$d_n := \begin{cases} x_{n+1} - \frac{1}{n} & x_{n+1} - c_n > 1, \\ x_{n+1} - \frac{x_{n+1} - c_n}{n} & x_{n+1} - c_n \leq 1. \end{cases}$$

Proof (i) Let

$$\chi_n(x) := \begin{cases} 1 & x \in [0, a+1], \\ 1 - \frac{1}{a_n} \int_{a+1}^{x} \frac{1}{\sigma\varphi}(t)dt & x \in [a+1, a+1+n], \\ 0 & \text{elsewhere}, \end{cases}$$

where

$$a_n = \int_{a+1}^{a+1+n} \frac{1}{\sigma\varphi}(s)ds.$$

Then $(\chi_n)_{n\geq 1} \subset \mathcal{F}$ with $0 \leq \chi_n \leq 1$, $n \geq 1$, $\chi_n \nearrow 1$ as $n \to \infty$ and $\lim_{n\to\infty} \mathcal{E}(\chi_n, \chi_n) = 0$.

(ii) Let

$$\chi_n(x) := \begin{cases} 1 & x \in [0, c_n], \\ 1 - \frac{1}{a_n}\int_{c_n}^{x} \frac{1}{\sigma\varphi}(t)dt & x \in [c_n, d_n], \\ 0 & \text{elsewhere}, \end{cases}$$

where

$$a_n = \int_{c_n}^{d_n} \frac{1}{\sigma\varphi}(s)ds.$$

Then $(\chi_n)_{n\geq 1} \subset \mathcal{F}$ satisfies the desired properties and determines recurrence of $(\mathcal{E}, \mathcal{F})$.

□

EXAMPLE **4.2** *If* $\varphi(x) = x^{\delta-1}$ *with* $\delta > 0$ *and* $\sigma(x) \equiv 1$ *on* I, *then clearly (4.18) is satisfied. In this case the process associated to the regular Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *(cf. [5]) is the well-known Bessel process of dimension* $\delta > 0$. *We are going to find a sufficient condition on the dimension* $\delta > 0$ *for recurrence. Since*

$$\int_1^\infty \frac{1}{\varphi}(s)ds = \int_1^\infty s^{1-\delta}ds = \begin{cases} \lim_{n\to\infty} \log n & \delta = 2, \\ \lim_{n\to\infty} \frac{1}{2-\delta}[n^{2-\delta} - 1] & \delta \neq 2, \end{cases}$$

we see by Theorem 4.4(i) with $a = 0$ *that the Bessel processes of dimension* $\delta > 0$ *is recurrent if* $\delta \in (0, 2]$. *Note that using [33, Theorem 3] we obtain the same calculations up to a constant. However, in [33] the Dirichlet form is supposed to be irreducible throughout which we do not demand.*

REMARK **4.8** *Of course Theorem 4.4 can be easily reformulated for Dirichlet forms defined on more general closed sets (cf. Remark 4.6).*

Chapter 5 Proofs of Lemmas 4.1, 4.2 and 4.3

Proof (of Lemma 4.1) Let $V \subset\subset E$. Then $(\mathcal{E}^0, D(\mathcal{E}^{0,V}))$ is a regular sectorial Dirichlet form on $L^2(V,\mu)$. Denote by $(L^{0,V}, D(L^{0,V}))$ its generator on $L^2(V,\mu)$. The following results can be derived similarly to in [36, Proposition 1.1, Theorem 1.5 and Lemma 1.6]. We obtain:

(i) $L^V u := L^{0,V} u + \langle B, \nabla u\rangle$, $u \in D(L^{0,V})_b$ is closable on $L^1(V,\mu)$. The closure $(\overline{L}^V, D(\overline{L}^V))$ is the generator of a sub-Markovian C_0-resolvent of contractions $(\overline{G}^V_\alpha)_{\alpha>0}$.

(ii) $D(\overline{L}^V)_b \subset D(\mathcal{E}^{0,V})$ and for $u \in D(\overline{L}^V)_b$, $v \in D(\mathcal{E}^{0,V})_b$, we have

$$\mathcal{E}^0(u,v) - \int_V \langle B, \nabla u\rangle v d\mu = -\int_V \overline{L}^V uvd\mu$$

and

$$\mathcal{E}^0(u,u) = -\int_V \overline{L}^V uud\mu.$$

Define for $f \in L^1(E,\mu)$, $\alpha > 0$

$$\overline{G}^V_\alpha f := \overline{G}^V_\alpha (f \cdot 1_V).$$

Then $(\overline{G}^V_\alpha)_{\alpha>0}$ can be extended to a sub-Markovian C_0-resolvent of contractions on $L^1(E,\mu)$. Indeed, let $(V_n)_{n\geq1}$ be relatively compact open sets in E such that $\overline{V}_n \subset V_{n+1}$ for all $n \geq 1$ and $\bigcup_{n\geq1} V_n = E$. In order to simplify the notations, let $(\mathcal{E}^{0,n}, D(\mathcal{E}^{0,n})) := (\mathcal{E}^{V_n}, D(\mathcal{E}^{V_n}))$ and $L^n := L^{V_n}$ for any $n \geq 1$. For $f \in L^1(E,\mu)$ with $f \geq 0$ μ-a.e. we let $\overline{G}^n_\alpha f := \overline{G}^{V_n}_\alpha f$. Then the following holds (cf. [36, Lemma

1.6])

$$\overline{G}_\alpha^n f \le \overline{G}_\alpha^{n+1} f,\ \mu\text{-a.e.} \tag{5.1}$$

Let $f \in L^1(E,\mu)_b$, with $f \ge 0$ μ-a.e., $\alpha > 0$ and $w_\alpha := \overline{G}_\alpha^n f - \overline{G}_\alpha^{n+1} f$. Then $w_\alpha \in D(\mathcal{E}^{0,n+1})_b$ and $w_\alpha^+ \in D(\mathcal{E}^{0,n})_b$. Since

$$\mathcal{E}^0(w_\alpha^+, w_\alpha^-) = \mathcal{E}^0(w_\alpha^+, w_\alpha^+ - w_\alpha) = -\mathcal{E}^0((-w_\alpha)\wedge 0, (-w_\alpha) - (-w_\alpha)\wedge 0) \le 0$$

and

$$\int_{V_{n+1}} \langle B, \nabla w_\alpha\rangle w_\alpha^+ d\mu = \int_{V_{n+1}} \langle B, \nabla w_\alpha^+\rangle w_\alpha^+ d\mu = 0,$$

we obtain

$$\begin{aligned}\mathcal{E}_\alpha^0(w_\alpha^+, w_\alpha^+) &\le \mathcal{E}_\alpha^0(w_\alpha, w_\alpha^+) - \int_{V_{n+1}} \langle B, \nabla w_\alpha\rangle w_\alpha^+ d\mu \\ &= \int_{V_{n+1}} (\alpha - L^n)\overline{G}_\alpha^n f w_\alpha^+ d\mu - \int_{V_{n+1}} (\alpha - L^{n+1})\overline{G}_\alpha^{n+1} f w_\alpha^+ d\mu = 0.\end{aligned}$$

Thus, $w_\alpha^+ = 0$ μ-a.e., i.e. $\overline{G}_\alpha^n f \le \overline{G}_\alpha^{n+1} f$ μ-a.e. Define for $f \in L^1(E,\mu)_b$, with $f \ge 0$ μ-a.e.

$$\overline{G}_\alpha f := \lim_{n\to\infty} \overline{G}_\alpha^n f.$$

Let $f \in L^1(E,\mu)$, $f \ge 0$ and $(f_n)_{n\ge 1} \subset L^1(E,\mu)_b$ with $0 \le f_n \le f_{n+1}$ μ-a.e. for any $n \ge 1$ be such that $f_n \to f$ in $L^2(E,\mu)$ as $n \to \infty$. Then

$$\overline{G}_\alpha f := \lim_{n\to\infty} \overline{G}_\alpha f_n$$

exists μ-a.e. since it is an increasing sequence. For general $f \in L^1(E,\mu)$, let $\overline{G}_\alpha f := \overline{G}_\alpha f^+ - \overline{G}_\alpha f^-$.

If $f \in L^1(E,\mu)$ with $f \ge 0$ μ-a.e., then

$$\overline{G}_\alpha f := \lim_{n\to\infty} \overline{G}_\alpha^n f$$

exists μ-a.e. and is independent of the choice of relatively open sets $(V_n)_{n\geq 1}$ by (5.1) and $(\overline{G}_\alpha)_{\alpha>0}$ is a sub-Markovian resolvent of contractions on $L^1(E,\mu)$ (cf. [36, Theorem 1.5(a)]). In order to show the strong continuity of $(\overline{G}_\alpha)_{\alpha>0}$, consider

$$u = \overline{G}^n_\alpha(\alpha - L^n)u = \overline{G}^n_\alpha(\alpha - L)u$$

for $u \in D(L^0)_{0,b}$ and $n \gg 1$. Hence, for $u \in D(L^0)_{0,b}$

$$u = \overline{G}_\alpha(\alpha - L)u.$$

Since $D(L^0)_{0,b} \subset L^1(E,\mu)$ densely (see, (C)) and

$$\begin{aligned}\|\alpha\overline{G}_\alpha u - u\|_{L^1(E,\mu)} &= \|\alpha\overline{G}_\alpha u - \overline{G}_\alpha(\alpha - L)u\|_{L^1(E,\mu)}\\ &= \|\overline{G}_\alpha Lu\|_{L^1(E,\mu)} \leq \frac{1}{\alpha}\|Lu\|_{L^1(E,\mu)} \longrightarrow 0\end{aligned}$$

as $\alpha \to \infty$, for any $u \in D(L^0)_{0,b}$, the strong continuity follows by a 3-ε-argument. Let $(\overline{L}, D(\overline{L}))$ be the generator of $(\overline{G}_\alpha)_{\alpha>0}$, then it satisfies conditions of Lemma 4.1 (see, [36, Theorem 1.5]).

□

Proof (of Lemma 4.2) Let $V \subset\subset E$. Since $\mathcal{E}^0_1$- and $\mathcal{E}^{0,h}_1$-norms are equivalent on $D(\mathcal{E}^{0,V})$, $(\mathcal{E}^{0,h}, D(\mathcal{E}^{0,V}))$ is also a regular sectorial Dirichlet form on $L^2(V,\mu)$. Denote by $(L^{0,h,V}, D(L^{0,h,V}))$ its generator on $L^2(V,\mu)$. Then we obtain $D(L^{0,V}) = D(L^{0,h,V})$ and $L^{0,h,V}u = L^{0,V}u - h\cdot u$ for $u \in D(L^{0,V}) = D(L^{0,h,V})$. Furthermore, similarly as in Lemma 4.1, we obtain:

(iii) $L^{h,V}u := L^{0,h,V}u + \langle B, \nabla u\rangle$, $u \in D(L^{0,h,V})_b$ is closable on $L^1(V,\mu)$. The closure $(\overline{L}^{h,V}, D(\overline{L}^{h,V}))$ is the generator of a sub-Markovian C_0-resolvent of contractions $(\overline{G}^{h,V}_\alpha)_{\alpha>0}$.

(iv) $D(\overline{L}^{h,V})_b \subset D(\mathcal{E}^{0,V})$ and for $u \in D(\overline{L}^{h,V})_b$, $v \in D(\mathcal{E}^{0,V})_b$, we have

$$\mathcal{E}^{0,h}(u,v) - \int_V \langle B, \nabla u\rangle v d\mu = -\int_V \overline{L}^{h,V} uv d\mu$$

and

$$\mathcal{E}^{0,h}(u,u) = -\int_V \overline{L}^{h,V} uu d\mu.$$

(v) $D(\overline{L}^{h,V}) = D(\overline{L}^{V})$ and for $u \in D(\overline{L}^{h,V})$,

$$\overline{L}^{h,V} u = \overline{L}^{V} u - h \cdot u.$$

Since the graph norms of $L^{h,V}$ and L^V are equivalent on $D(L^{0,V})$, we obtain the last statement (v).

Define for $f \in L^1(E,\mu)$, $\alpha > 0$

$$\overline{G}_\alpha^{h,V} f := \overline{G}_\alpha^{h,V}(f \cdot 1_V).$$

Then similarly to the above proof of Lemma 4.1, $(\overline{G}_\alpha^{h,V})_{\alpha>0}$ can be extended to a sub-Markovian C_0-resolvent of contractions on $L^1(E,\mu)$. As in the $(\overline{G}_\alpha)_{\alpha>0}$ case, choose relatively compact open sets $(V_n)_{n\geq 1}$ such that $\overline{V}_n \subset V_{n+1}$ for all $n \geq 1$ and $\bigcup_{n\geq 1} V_n = E$ and let $\overline{G}_\alpha^{h,n} := \overline{G}_\alpha^{h,V_n}$ and $(\overline{L}^{h,n}, D(\overline{L}^{h,n})) := (\overline{L}^{h,V_n}, D(\overline{L}^{h,V_n}))$, $n \geq 1$. Then for $f \in L^1(E,\mu)$ with $f \geq 0$ μ-a.e., $\overline{G}_\alpha^{h,n} f$ is increasing in n and

$$\overline{G}_\alpha^{h} f := \lim_{n\to\infty} \overline{G}_\alpha^{h,n} f$$

exists μ-a.e. and is independent of the choice of relatively compact open sets $(V_n)_{n\geq 1}$. For general $f \in L^1(E,\mu)$, let $\overline{G}_\alpha^{h} f := \overline{G}_\alpha^{h} f^+ - \overline{G}_\alpha^{h} f^-$. Then $(\overline{G}_\alpha^{h})_{\alpha>0}$ is a sub-Markovian C_0-resolvent of contractions on $L^1(E,\mu)$ and its generator $(\overline{L}^{h}, D(\overline{L}^{h}))$ satisfies properties (i) and (ii) of Lemma 4.2.

Next, we show that $D(\overline{L}^h) = D(\overline{L})$. By definition, if $u \in D(\overline{L})$, then there exists $f \in L^1(E,\mu)$ such that

$$u = \overline{G}_\alpha f = \lim_{n\to\infty} \overline{G}^n_\alpha(f \cdot 1_{V_n})$$

where $V_n \subset\subset E$, $\overline{V}_n \subset V_{n+1}$ and $V_n \nearrow E$ as $n \to \infty$. Without loss of generality, we may assume that $f \geq 0$. Since V_n is relatively compact open for $n \geq 1$, $D(\overline{L}^n) = D(\overline{L}^{h,n})$ by (v). So there exists a sequence of functions $(g_n)_{n\geq 1} \subset L^1(E,\mu)$ with $\mathrm{supp}(g_n) \subset \overline{V}_n$ such that

$$\overline{G}^n_\alpha(f \cdot 1_{V_n}) = \overline{G}^{h,n}_\alpha g_n$$

for all $n \geq 1$. In particular, $(\alpha - \overline{L}^{h,n})\overline{G}^n_\alpha(f \cdot 1_{V_n}) = g_n$. Since $-\overline{L}^{h,n} u = -\overline{L}^n u + h \cdot u$ for any $u \in D(\overline{L}^{h,n}) = D(\overline{L}^n)$,

$$g_n = f \cdot 1_{V_n} + h \cdot \overline{G}^n_\alpha(f \cdot 1_{V_n}) \geq 0.$$

Since $\overline{G}^n_\alpha(f \cdot 1_{V_n})$ is increasing in n and converges to $\overline{G}_\alpha f$ μ-a.e. and in $L^1(E,\mu)$, g:=$\lim_{n\to\infty} g_n = f + h \cdot \overline{G}_\alpha f$ exists μ-a.e.

We claim that

$$\lim_{n\to\infty} \overline{G}^{h,n}_\alpha(g \cdot 1_{V_n}) = \lim_{n\to\infty} \overline{G}^{h,n}_\alpha g_n.$$

Since $\overline{G}^n_\alpha(f 1_{V_n}) \leq \overline{G}_\alpha f$, we have $g_n \leq g 1_{V_n}$, hence $\overline{G}^{h,n}_\alpha g_n \leq \overline{G}^{h,n}_\alpha(g \cdot 1_{V_n})$ and

$$\begin{aligned}\|\overline{G}^{h,n}_\alpha(g \cdot 1_{V_n}) - \overline{G}^{h,n}_\alpha g_n\|_{L^1(E,\mu)} &= \left\|\overline{G}^{h,n}_\alpha\left[1_{V_n} h \left(\overline{G}_\alpha f - \overline{G}^n_\alpha(f \cdot 1_{V_n})\right)\right]\right\|_{L^1(E,\mu)} \\ &\leq \frac{1}{\alpha}\|h\|_{L^\infty(E,\mu)}\|\overline{G}_\alpha f - \overline{G}^n_\alpha(f \cdot 1_{V_n})\|_{L^1(E,\mu)}.\end{aligned}$$

Since $\lim_{n\to\infty} \overline{G}^n_\alpha(f \cdot 1_{V_n}) = \overline{G}_\alpha f$ in $L^1(E,\mu)$ and $\lim_{n\to\infty} \overline{G}^{h,n}_\alpha(g \cdot 1_{V_n}) = \overline{G}^h_\alpha g$ in $L^1(E,\mu)$, we have $\lim_{n\to\infty} \overline{G}^{h,n}_\alpha g_n = \overline{G}^h_\alpha g$ in $L^1(E,\mu)$. Since $\overline{G}^{h,n}_\alpha g_n$ is increasing in n, it converges μ-a.e. as $n \to \infty$ hence, $\lim_{n\to\infty} \overline{G}^{h,n}_\alpha g_n = \overline{G}^h_\alpha g$ μ-a.e. and in

$L^1(E,\mu)$. Therefore,

$$\overline{G}_\alpha f = \lim_{n\to\infty} \overline{G}_\alpha^{h,n} g_n = \lim_{n\to\infty} \overline{G}_\alpha^{h,n}(g\cdot 1_{V_n}) = \overline{G}_\alpha^h g$$

with $g = f + h\cdot \overline{G}_\alpha f$ and so $D(\overline{L}) \subset D(\overline{L}^h)$.

Likewise, if $u \in D(\overline{L}^h)$ is such that $u = \overline{G}_\alpha^h f = \lim_{n\to\infty} \overline{G}_\alpha^{h,n}(f\cdot 1_{V_n})$ where $f \in L^1(E,\mu)$ with $f \geq 0$, then $u \in D(\overline{L})$ and $\overline{G}_\alpha^h f = \overline{G}_\alpha g$, where $g = f - h\cdot \overline{G}_\alpha^h f$.

□

Proof (of Lemma 4.3) Consider the real Hilbert space $L^2(E,(h+\varepsilon)\mu)$. Then it is easy to see that $(\mathcal{E}^{0,V}, D(\mathcal{E}^{0,V}))$ is a regular sectorial Dirichlet form on $L^2(V,(h+\varepsilon)\mu)$. Denote by $(L^{0,\varepsilon,V}, D(L^{0,\varepsilon,V}))$ the $L^2(V,(h+\varepsilon)\mu)$-generator of $(\mathcal{E}^{0,V}, D(\mathcal{E}^{0,V}))$ on $L^2(V,(h+\varepsilon)\mu)$. Then we can show $D(L^{0,\varepsilon,V}) = D(L^{0,V})$. Furthermore, we obtain:

(vi) $L^{\varepsilon,V}u := L^{0,\varepsilon,V}u + \langle B^\varepsilon, \nabla u\rangle$, $u \in D(L^{0,\varepsilon,V})_b$ is closable on $L^1(V,(h+\varepsilon)\mu)$. The closure $(\overline{L}^{\varepsilon,V}, D(\overline{L}^{\varepsilon,V}))$ is the generator of a sub-Markovian C_0-resolvent of contractions $(\overline{G}_\alpha^{\varepsilon,V})_{\alpha>0}$.

(vii) $D(\overline{L}^{\varepsilon,V})_b \subset D(\mathcal{E}^{0,\varepsilon,V})$ and for $u \in D(\overline{L}^{\varepsilon,V})_b$, $v \in D(\mathcal{E}^{0,\varepsilon,V})_b$, we have

$$\mathcal{E}^0(u,v) - \int_V \langle B^\varepsilon, \nabla u\rangle v(h+\varepsilon)d\mu = -\int_V \overline{L}^{\varepsilon,V} uv(h+\varepsilon)d\mu$$

and

$$\mathcal{E}^0(u,u) = -\int_V \overline{L}^{\varepsilon,V} uu(h+\varepsilon)d\mu.$$

(viii) $D(\overline{L}^{\varepsilon,V}) = D(\overline{L}^V)$ and for $u \in D(\overline{L}^{\varepsilon,V})$,

$$\overline{L}^{\varepsilon,V} u = \frac{1}{h+\varepsilon}\overline{L}^V u.$$

Define for $f \in L^1(E,(h+\varepsilon)\mu)$, $\alpha > 0$,

$$\overline{G}_\alpha^{\varepsilon,V} f := \overline{G}_\alpha^{\varepsilon,V}(f \cdot 1_V),$$

then $(\overline{G}_\alpha^{\varepsilon,V})_{\alpha>0}$ can be extended to a sub-Markovian C_0-resolvent of contractions on $L^1(E,(h+\varepsilon)\mu)$. As in the $(\overline{G}_\alpha)_{\alpha>0}$ case, choose a sequence of relatively compact open sets $(V_n)_{n\geq 1}$ such that $\overline{V}_n \subset V_{n+1}$ for all $n \geq 1$ and $\bigcup_{n\geq 1} V_n = E$. As in the proof of Lemma 4.2, let $\overline{G}_\alpha^{\varepsilon,n} := \overline{G}_\alpha^{\varepsilon,V_n}$ and $(\overline{L}^{\varepsilon,n}, D(\overline{L}^{\varepsilon,n})) := (\overline{L}^{\varepsilon,V_n}, D(\overline{L}^{\varepsilon,V_n}))$ for $n \geq 1$. Then for $f \in L^1(E,(h+\varepsilon)\mu)$ with $f \geq 0$ μ-a.e., $\overline{G}_\alpha^{\varepsilon,n} f$ is increasing in n and

$$\overline{G}_\alpha^{\varepsilon} f := \lim_{n\to\infty} \overline{G}_\alpha^{\varepsilon,n} f$$

exists μ-a.e. and is independent of the choice of relatively compact open sets $(V_n)_{n\geq 1}$. For general $f \in L^1(E,(h+\varepsilon)\mu)$, let $\overline{G}_\alpha^{\varepsilon} f := \overline{G}_\alpha^{\varepsilon} f^+ - \overline{G}_\alpha^{\varepsilon} f^-$. Then $(\overline{G}_\alpha^{\varepsilon})_{\alpha>0}$ is a sub-Markovian C_0-resolvent of contractions on $L^1(E,(h+\varepsilon)\mu)$ and its generator $(\overline{L}^{\varepsilon}, D(\overline{L}^{\varepsilon}))$ satisfies properties (i) and (ii) of Lemma 4.3.

Next we show $D(\overline{L}^{\varepsilon}) = D(\overline{L})$. If $u \in D(\overline{L}^{\varepsilon})$, then there exists $f \in L^1(E,(h+\varepsilon)\mu)$ such that $u = \overline{G}_\alpha^{\varepsilon} f = \lim_{n\to\infty} \overline{G}_\alpha^{\varepsilon,n}(f \cdot 1_{V_n})$ where $V_n \subset\subset E$ and $V_n \nearrow E$ as $n \to \infty$. Without loss of generality, we may assume that $f \geq 0$. Since V_n is relatively compact open in E for all $n \geq 1$, $D(\overline{L}^{\varepsilon,n}) = D(\overline{L}^n)$. So there exists a sequence of functions $(g_n)_{n\geq 1} \subset L^1(E,\mu)$ with $\operatorname{supp}(g_n) \subset \overline{V}_n$ such that

$$\overline{G}_\alpha^{\varepsilon,n}(f \cdot 1_{V_n}) = \overline{G}_\alpha^n g_n$$

for all $n \geq 1$. We have $(\alpha - \overline{L}^n)\overline{G}_\alpha^{\varepsilon,n}(f \cdot 1_{V_n}) = g_n$. Since $\overline{L}^{\varepsilon,n} u = \frac{1}{h+\varepsilon}\overline{L}^n u$ for $u \in D(\overline{L}^n) = D(\overline{L}^{\varepsilon,n})$,

$$g_n = (h+\varepsilon) f \cdot 1_{V_n} + \alpha(1-(h+\varepsilon))\overline{G}_\alpha^{\varepsilon,n}(f \cdot 1_{V_n}).$$

Since $\lim_{n\to\infty} \overline{G}_\alpha^{\varepsilon,n}(f \cdot 1_{V_n}) = \overline{G}_\alpha^\varepsilon f$ $(h+\varepsilon)\mu$-a.e. and in $L^1(E,(h+\varepsilon)\mu)$,

$$g := \lim_{n\to\infty} g_n = (h+\varepsilon)f + \alpha(1-(h+\varepsilon))\overline{G}_\alpha^\varepsilon f$$

exists μ-a.e. We claim that

$$\lim_{n\to\infty} \overline{G}_\alpha^n g_n = \lim_{n\to\infty} \overline{G}_\alpha^n (g \cdot 1_{V_n}).$$

Indeed,

$$\begin{aligned}\|\overline{G}_\alpha^n(g\cdot 1_{V_n}) - \overline{G}_\alpha^n g_n\|_{L^1(E,\mu)} &\le (1+\|h\|_{L^\infty(E,\mu)}+\varepsilon)\cdot \left\|\overline{G}_\alpha^\varepsilon f - \overline{G}_\alpha^{\varepsilon,n}(f\cdot 1_{V_n})\right\|_{L^1(E,\mu)} \\ &\le \frac{1}{\varepsilon}(1+\|h\|_{L^\infty(E,\mu)}+\varepsilon)\cdot \left\|\overline{G}_\alpha^\varepsilon f - \overline{G}_\alpha^{\varepsilon,n}(f\cdot 1_{V_n})\right\|_{L^1(E,(h+\varepsilon)\mu)}\end{aligned}$$

and since $\lim_{n\to\infty} \overline{G}_\alpha^{\varepsilon,n}(f \cdot 1_{V_n}) = \overline{G}_\alpha^\varepsilon f$ in $L^1(E,(h+\varepsilon)\mu)$ and $\lim_{n\to\infty} \overline{G}_\alpha^n(g \cdot 1_{V_n}) = \overline{G}_\alpha g$ in $L^1(E,\mu)$, we have $\lim_{n\to\infty} \overline{G}_\alpha^n g_n = \overline{G}_\alpha g$ in $L^1(E,\mu)$. Since $\overline{G}_\alpha^n g_n$ is increasing in n, it converges μ-a.e. as $n \to \infty$. Moreover, $\lim_{n\to\infty} \overline{G}_\alpha^n g_n = \overline{G}_\alpha g$ μ-a.e. and in $L^1(E,\mu)$. Therefore,

$$\overline{G}_\alpha^\varepsilon f = \lim_{n\to\infty} \overline{G}_\alpha^{\varepsilon,n}(f\cdot 1_{V_n}) = \lim_{n\to\infty} \overline{G}_\alpha^n(g\cdot 1_{V_n}) = \overline{G}_\alpha g.$$

Likewise, we can show converse that if $f \in D(\overline{L})$, then there exists a function $g \in L^1(E,(h+\varepsilon)\mu)$ such that $\overline{G}_\alpha f = \overline{G}_\alpha^\varepsilon g$.

□

Part II

Conservativeness criteria for generalized Dirichlet forms

Chapter 6 A general criterion for conservativeness of a generalized Dirichlet form

In this Chapter, we characterize conservativeness analytically in the non-sectorial case and derive an analytic conservative criterion for a generalized Dirichlet form $\mathcal{E}$ which is expressed as

$$\mathcal{E}(f,g) = \mathcal{E}^0(f,g) + \int_E fNg d\mu,$$

where $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a symmetric strongly local regular Dirichlet form on $L^2(E,\mu)$ which is represented by a carré du champ $\mathcal{E}^0(f,g) = \int_E \Gamma(f,g)d\mu$ and $(N, D(N))$ is a linear operator on $L^2(E,\mu)$ (see (C2) below).

Let us recall the definition of conservative (see, Definition 4.1): $(T_t)_{t>0}$ is said to be conservative if

$$T_t 1 = 1 \ \mu\text{-a.e. for some (and hence any) } t > 0. \tag{6.1}$$

LEMMA **6.1** *Let D be an arbitrary dense subset of $L^1(E,\mu)$. Then, $(T_t)_{t>0}$ is conservative, if and only if for some (and hence any) $t>0$*

$$\int_E \widehat{T}_t f d\mu = \int_E f d\mu \quad \textit{for any } f \in D, \tag{6.2}$$

i.e. μ is $(\widehat{T}_t)_{t>0}$-invariant.

Proof Since the first statement is obvious, we only show that if (6.2) (hence equivalently (6.1)) holds for some $t>0$, then it holds for all $t>0$. Assume hence that

$$T_t 1 = 1 \ \mu\text{-a.e. for some } t > 0.$$

Let $(f_n)_{n\geq 1} \subset L^2(E,\mu) \cap L^\infty(E,\mu)$, $0 \leq f_n \nearrow 1$ as $n \to \infty$. Then by definition we obtain

$$\lim_{n\to\infty} T_t f_n = 1, \ \ \mu\text{-a.e.}$$

Let $t, s > 0$. Since $T_{t+s} f_n = T_t(T_s f_n)$, it suffices to show that

$$\lim_{n\to\infty} T_s f_n = T_s 1 = 1, \ \ \mu\text{-a.e.}$$

for any $0 < s < t$. Let $0 < s < t$ and suppose that we do not have

$$\lim_{n\to\infty} T_s f_n = 1, \ \ \mu\text{-a.e.}$$

Then there exists a measurable set A with $0 < \mu(A) < \infty$ such that

$$\lim_{n\to\infty} T_s f_n < 1 \ \mu\text{-a.e. on } A.$$

Since $(\widehat{T}_t)_{t>0}$ is an $L^1(E,\mu)$ contraction,

$$\int_E \widehat{T}_t 1_A d\mu \leq \int_E \widehat{T}_s 1_A d\mu = \lim_{n\to\infty} \int_E \widehat{T}_s 1_A f_n d\mu = \lim_{n\to\infty} \int_A T_s f_n d\mu < \int_E 1_A d\mu$$

which leads to the contradiction.

□

Fix $t > 0$. From now on until the end of Chapter 6, we assume:

(C1) Let $(V_n)_{n\geq 1}$ be an arbitrary increasing sequence of relatively compact open sets in E such that $\overline{V}_n \subset V_{n+1}$ and $\bigcup_{n\geq 1} V_n = E$. Then for $p = 1$ or $p = 2$, there exist sub-Markovian C_0-semigroups of contractions $(\widehat{T}^n_t)_{t>0}$,

$n \geq 1$ on $L^p(V_n, \mu)$ with generators $(\widehat{L}^n, D(\widehat{L}^n))$, $n \geq 1$, such that for any non-negative $f \in L^1(E,\mu) \cap L^\infty(E,\mu)$,

$$\widehat{T}_t^n f := \widehat{T}_t^n (f \cdot 1_{V_n}) \nearrow \widehat{T}_t f \ \mu\text{-a.e. as } n \to \infty.$$

Next, we aim to give a general criterion for conservativeness in case the generalized Dirichlet form can be represented locally by a linear perturbation of a symmetric strongly local regular Dirichlet form. By the latter, we mean that there exists a symmetric strongly local regular Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ on $L^2(E,\mu)$ in the sense of [5, I.1.1], expressed as

$$\mathcal{E}^0(u,v) = \int_E \Gamma(u,v)(x)\mu(dx), \text{ for } u,v \in D(\mathcal{E}^0),$$

where Γ is a positive semidefinite symmetric bilinear form on $D(\mathcal{E}^0)$ with values in $L^1(E,\mu)$ (see [3]) such that for each $n \geq 1$, $D(\widehat{L}^n)_b \subset D(\mathcal{E}^0)_b$ and there exists a linear operator $N : D(N) \longrightarrow L^1(E,\mu)_{loc}$ on $L^2(E,\mu)$ with $D(\mathcal{E}^0)_b \subset D(N)$ such that

$$-\int_{V_n} \widehat{L}^n v u d\mu = \int_E \Gamma(u,v) d\mu + \int_{V_n} u N v d\mu \tag{6.3}$$

for any $v \in D(\widehat{L}^n)_b$ and $u \in D(\mathcal{E}^0)$ and $D(N)$ contains $u \cdot 1_{V_n}$ where $u \in D(\mathcal{E}^0)_{loc,b}$. Here the term strongly local means that $\mathcal{E}^0(u,v) = 0$ whenever u is a constant on a neighborhood of $\text{supp}(v)$. The linear operator $(N, D(N))$ needs not to be a generator of a C_0-semigroup of contractions on $L^2(E,\mu)$ but satisfies

$$v \in D(\mathcal{E}^0)_b, \ v = \text{constant } \mu\text{-a.e. on } B \in \mathcal{B}(E) \text{ implies } Nv = 0 \ \mu\text{-a.e. on } B, \tag{6.4}$$

$$\int_E N v d\mu = 0 \text{ for any } v \in D(\mathcal{E}^0)_{0,b} \tag{6.5}$$

and

$$N\phi(v) = \phi'(v)Nv,\ \ N(uv) = vNu + uNv \text{ for any } u, v \in D(N)_b,\ \ \phi \in C_b^1(\mathbb{R}). \tag{6.6}$$

Thus, from now on until the end of Chapter 6, we assume that the following condition holds:

(C2) for each $n \geq 1$, $(\widehat{L}^n, D(\widehat{L}^n))$ can be represented as in (6.3) and $(N, D(N))$ satisfies (6.4), (6.5) and (6.6).

For the convenience of the reader, we recall here some basic properties of strongly local regular Dirichlet forms, which can be represented by a carré du champ. For any $u \in D(\mathcal{E}^0)$, there is a unique finite measure $\mu_{\langle u\rangle}$ on E called the energy measure of u such that

$$\int_E d\mu_{\langle u\rangle} = 2\mathcal{E}^0(u,u)$$

and if $u \in D(\mathcal{E}^0)_b$, then we get

$$\int_E f d\mu_{\langle u\rangle} = 2\mathcal{E}^0(u, fu) - \mathcal{E}^0(u^2, f),$$

for any $f \in C_b(E) \cap D(\mathcal{E}^0)$. Then $\mu_{\langle u,v\rangle}$, $u, v \in D(\mathcal{E}^0)$ is defined by polarization, i.e.

$$\mu_{\langle u,v\rangle} := \frac{1}{2}\left(\mu_{\langle u+v\rangle} - \mu_{\langle u\rangle} - \mu_{\langle v\rangle}\right)$$

Since $\mu_{\langle u,v\rangle}$ is bilinear in u, v and $\mu_{\langle u\rangle}$ is positive, we obtain for non-negative $f \in C_b(E) \cap D(\mathcal{E}^0)$

$$\left|\left(\int_E f\mu_{\langle u\rangle}\right)^{1/2} - \left(\int_E f\mu_{\langle v\rangle}\right)^{1/2}\right| \leq \left(\int_E f d\mu_{\langle u-v\rangle}\right)^{1/2}.$$

The energy measure then satisfies for any $u, v \in D(\mathcal{E}^0)$

$$\int_E f d\mu_{\langle u,v\rangle} = 2\int_E f\Gamma(u,v)d\mu, \quad f \in C_b(E) \cap D(\mathcal{E}^0).$$

Since $(\mathcal{E}^0, D(\mathcal{E}^0))$ is strongly local, the energy measures $\mu_{\langle u,v\rangle}$, $u, v \in D(\mathcal{E}^0)$, are strongly local and satisfy the Leibniz and the Chain rules. In particular, $\mu_{\langle u\rangle}$ can be extended to $u \in D(\mathcal{E}^0)_{loc}$ and $\Gamma(u,v)$ satisfies the Leibniz and Chain rules (see [5] and [33]).

We assume from now on until the end of Chapter 6 that

(C3) there exists a non-negative continuous function ρ on E with

$$\rho \in D(\mathcal{E}^0)_{loc}$$

such that for $r > 0$

$$E_r := \{x \in E : \rho(x) < r\}$$

is a relatively compact open set in E and $\bigcup_{r>0} E_r = E$. Furthermore, there exists a compact subset K_0 of E such that

$$\Gamma(\rho,\rho),\ N(\rho) \in L^\infty_{loc}(K_0^c, \mu).$$

REMARK **6.1** *Let $(\mathcal{E}, \mathcal{F})$ be a symmetric strongly local and regular Dirichlet form. Then we may define the part Dirichlet forms $(\mathcal{E}^n, \mathcal{F}^n)$ corresponding to an increasing sequence of relatively compact open sets $(V_n)_{n\geq 1}$ such that $\bigcup_{n\geq 1} V_n = E$ where $\mathcal{F}^n = \{u \in \mathcal{F} : \widetilde{u} = 0$ q.e. on $V_n^c\}$ and $\widetilde{u}$ is a quasi continuous version of $u \in \mathcal{F}$ (see [5, Theorem 4.4.5]), i.e. $\widetilde{u} = 0$ up to a capacity zero set on V_n^c. Denote the associated semigroups of $(\mathcal{E}^n, \mathcal{F}^n)$ by $(T_t^n)_{t>0}$ and the associated linear operators by $(L^n, D(L^n))$ on $L^2(V_n, \mu)$. Then, obviously $(T_t^n)_{t>0}$ and $(L^n, D(L^n))$ are symmetric. We know that*

$$T_t f = \lim_{n\to\infty} T_t^n f \ \mu\text{-}a.e.$$

for any $f \in L^2(E,\mu)$ *where* $T_t^n f := T_t^n(f \cdot 1_{V_n})$. *In particular, if* f *is non-negative, then* $T_t^n f \nearrow T_t f$ μ*-a.e. as* $n \to \infty$. *Moreover, as explained before* $(\mathcal{E},\mathcal{F})$ *can be represented by a carré du champ. Thus,* $(\mathcal{E},\mathcal{F})$ *satisfies (C1) with* $p=2$. *Furthermore, for* $v \in D(L^n)$,

$$\mathcal{E}(u,v) = (-L^n v, u), \quad \textit{for any } u \in \mathcal{F}$$

which implies that (6.3) holds. Putting $N \equiv 0$ *implies that (C2) holds. Moreover, if the topology induced by the intrinsic metric* d^{int} *defined by*

$$d^{int}(x,y) := \sup\{u(x) - u(y) : u \in \mathcal{F}_{loc} \cap C(E),\ \Gamma(u,u) \le 1 \textit{ on } E\}$$

introduced in [33] is equivalent to the original topology on E *and any balls induced by the intrinsic metric are relatively compact open sets, then we may choose* $\rho(x) := d^{int}(x,x_0)$ *for some fixed* $x_0 \in E$ *(see [33, Lemma 1]). Hence (C3) holds.*

By the assumption (C3),

$$V_n := E_{4n},\ n \ge 1, \tag{6.7}$$

are relatively compact open subsets of E with $\bigcup_{n\ge1} V_n = E$. From now on fix $(V_n)_{n\ge1}$ as in (6.7) and note that (C1) and (C2) hold for this choice of $(V_n)_{n\ge1}$. For a function f which has compact support, define

$$k_f := \min\{m \in \mathbb{N} : \mathrm{supp}(f) \subset E_m \text{ and } K \subset E_m\}, \tag{6.8}$$

where K is an arbitrary but fixed compact subset of E containing K_0 as in (C3). Let

$$D_0 := \{f : f \in L^\infty(E,\mu) \cap L^2(E,\mu)_0 \text{ such that } \widehat{T}_s^n f \in D(\widehat{L}^n) \text{ for any } n \ge k_f,\ s \in [0,t]\}. \tag{6.9}$$

In order to perform comfortably our calculations up to the formulation and proof of Theorem 6.1 below, we do need the following auxiliary assumption

(A) there exists $f \in D_0$ such that $\mathrm{supp}(f) \neq \emptyset$.

REMARK **6.2** *Assumption (A) will be replaced by the stronger (C4) occurring right after the proof of Theorem 6.1 below. Note that if the* $(\widehat{T}_t^n)_{t>0}$*,* $n \geq 1$ *are analytic, then* $\widehat{T}_s^n f \in D(\widehat{L}^n)$ *for any* $f \in L^1(E,\mu) \cap L^2(E,\mu)$*. Thus, (A) and (C4) below trivially hold. In the non-sectorial (i.e. non-analytic) case, we can impose the reasonable assumption that the coefficients of the generators of* $(\widehat{T}_t^n)_{t>0}$*,* $n \geq 1$*, are p-fold integrable with respect to the measure* μ*, where p is either 1 or 2 (as in (C)). Then* $C_0^\infty(E) \subset D_0$ *for instance in the case where* $E := \mathbb{R}^d$ *and there are no boundary conditions (cf. Subsections 8.2.1 and 8.2.2). In particular, (C4) below is then also automatically satisfied. Similarly, one can easily obtain nice dense subsets of* D_0 *in case of boundary conditions provided the coefficients are not too singular. To keep this exposition reasonably sized and because of the similarity to the case without boundary conditions, we didn't include an example.*

LEMMA **6.2** *Let* $D \subset D_0$ *be an arbitrary dense subset of* $L^1(E,\mu)$*. Then* $(T_t)_{t>0}$ *is conservative, if and only if there exists a sequence of functions* $(\chi_n)_{n\geq 1} \subset L^2(V_n,\mu)$ *such that* $0 \leq \chi_n \nearrow 1$ *as* $n \to \infty$ *and*

$$\lim_{n\to\infty} \left[\int_0^t \int_E \frac{d}{ds} \widehat{T}_s^n f \cdot \chi_n \, d\mu ds \right] = 0$$

for any $f \in D$ *and some (and hence all)* $t > 0$*.*

Proof Let $f \in D$ and $(\chi_n)_{n\geq 1}$ be as in the statement. Then by (C1),

$$\int_E \left(\widehat{T}_t f - f\right) d\mu = \lim_{n\to\infty} \int_E (\widehat{T}_t^n f - f)\chi_n d\mu = \lim_{n\to\infty} \left[\int_0^t \int_E \frac{d}{ds}\widehat{T}_s^n f \cdot \chi_n d\mu ds\right] \tag{6.10}$$

for any $f \in D$ and the assertion follows by Lemma 6.1.

□

Now we are looking for a more explicit criterion for conservativeness of $(T_t)_{t>0}$. From now on unless otherwise stated, let us fix f as in (A). Let for $n \geq 1$,

$$\chi_n(x) := 1 \wedge \left(2 - \frac{\rho(x)}{2n}\right)^+ \tag{6.11}$$

and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ in $C^1(\mathbb{R}^+)$ be increasing and such that $\phi(0) = 0$ and $\phi(r) \nearrow +\infty$ as $r \nearrow +\infty$. Then for each $n \geq 1$,

$$\psi_n(x) := (\phi(\rho(x)) - \phi(k_f))^+ \wedge (\phi(4n) - \phi(k_f))^+ \in D(\mathcal{E}^0)_{loc}. \tag{6.12}$$

Note that $(\chi_n)_{n\geq 1} \subset D(\mathcal{E}^0)_{0,b}$ by (C3). Now we will use the method of Davies, Oshima and Uemura. Let

$$\widehat{T}_s^{\psi_n} f := e^{\psi_n}\widehat{T}_s^n(f e^{-\psi_n}). \tag{6.13}$$

Then $\widehat{T}_s^{\psi_n} f \in D(\mathcal{E}^0) \cap L^\infty(V_n, \mu)$ with $\widehat{T}_s^{\psi_n} f = 0$ on V_n^c for any $s > 0$, and $\widehat{T}_s^{\psi_n} f = e^{\psi_n}\widehat{T}_s^n f$ for any $n \geq 1$ because $\psi_n \equiv 0$ on E_{k_f} for any $n \geq 1$. For $t > 0$, let

$$\widehat{v}_t := \int_0^t \widehat{T}_s^{\psi_n} f ds.$$

Let $n \geq k_f$. By Leibniz and Chain rules for Γ and N, (6.4) and Fubini, we obtain that

$$
\begin{aligned}
&\left|\int_0^t \int_E \frac{d}{ds}\widehat{T}_s^n f \cdot \chi_n d\mu ds\right| = \left|\int_{V_n} \widehat{L}^n \left(\int_0^t \widehat{T}_s^n f ds\right) \cdot \chi_n d\mu\right| \\
&= \left|-\int_{V_n} \Gamma\left(\chi_n, e^{-\psi_n}\widehat{v}_t\right) d\mu - \int_{V_n} \chi_n N\left(e^{-\psi_n}\widehat{v}_t\right) d\mu\right| \\
&= \left|\int_{V_n} \left(\Gamma(\chi_n, \psi_n) + N(\chi_n)\right) e^{-\psi_n}\widehat{v}_t d\mu - \int_{V_n} \Gamma\left(\chi_n, \widehat{v}_t\right) e^{-\psi_n} d\mu\right| \\
&\leq \frac{e^{\phi(k_f)-\phi(2n)}}{2n}\left[\int_{E_{4n}\setminus E_{2n}} \left|\phi'(\rho)\Gamma(\rho,\rho) + N(\rho)\right| \cdot |\widehat{v}_t| d\mu + \left|\int_{E_{4n}\setminus E_{2n}} \Gamma\left(\rho, \widehat{v}_t\right) d\mu\right|\right] \\
&\leq \frac{e^{\phi(k_f)-\phi(2n)}}{2n}\Bigg[\left\{\left(\int_{E_{4n}\setminus E_{2n}} \left(\phi'(\rho)\Gamma(\rho,\rho)\right)^2 d\mu\right)^{1/2} + \left(\int_{E_{4n}\setminus E_{2n}} \left(N(\rho)\right)^2 d\mu\right)^{1/2}\right\} \|\widehat{v}_t\|_{L^2(V_n,\mu)} \\
&\qquad\qquad + \left(\int_{E_{4n}\setminus E_{2n}} \Gamma(\rho,\rho) d\mu\right)^{1/2} \left(\int_{V_n} \Gamma\left(\widehat{v}_t, \widehat{v}_t\right) d\mu\right)^{1/2}\Bigg] \\
&\leq \frac{e^{\phi(k_f)-\phi(2n)}}{2n}\Bigg[\left(\operatorname*{ess\,sup}_{E_{4n}\setminus E_{2n}} \left(\phi'(\rho)\Gamma(\rho,\rho)\right) \mu(E_{4n}\setminus E_{2n})^{1/2} + \|N(\rho)\|_{L^2(E_{4n}\setminus E_{2n},\mu)}\right) \|\widehat{v}_t\|_{L^2(V_n,\mu)} \\
&\qquad\qquad + \operatorname*{ess\,sup}_{E_{4n}\setminus E_{2n}} \Gamma(\rho,\rho)^{1/2} \mu(E_{4n}\setminus E_{2n})^{1/2} \left(\int_{V_n} \Gamma\left(\widehat{v}_t, \widehat{v}_t\right) d\mu\right)^{1/2}\Bigg] \\
&\leq \frac{e^{\phi(k_f)-\phi(2n)}}{2n}\Big(\mu(E_{4n}\setminus E_{2n})^{1/2} \left(\sqrt{a_n}\mathcal{E}^0\left(\widehat{v}_t, \widehat{v}_t\right)^{1/2} + b_n\|\widehat{v}_t\|_{L^2(V_n,\mu)}\right) \\
&\qquad\qquad + \|N(\rho)\|_{L^2(E_{4n}\setminus E_{2n},\mu)} \|\widehat{v}_t\|_{L^2(V_n,\mu)}\Big)
\end{aligned} \tag{6.14}
$$

where

$$a_n := \operatorname*{ess\,sup}_{E_{4n}\setminus E_{2n}} \Gamma(\rho,\rho) \tag{6.15}$$

and

$$b_n := \operatorname*{ess\,sup}_{E_{4n}\setminus E_{2n}} \phi'(\rho)\Gamma(\rho,\rho). \tag{6.16}$$

Since Γ is positive semidefinite and ϕ is increasing, a_n and b_n are nonnegative and well-defined by (C3) and (6.8). Now, we are going to find the following estimates in (6.14)

$$\|\widehat{v}_t\|_{L^2(V_n,\mu)} \leq te^{c_n(f)t}\|f\|_{L^2(E,\mu)}$$

and

$$\mathcal{E}^0\left(\widehat{v}_t, \widehat{v}_t\right)^{1/2} \le \sqrt{3t} e^{c_n(f)t} \|f\|_{L^2(E,\mu)}$$

where

$$c_n(f) := \operatorname*{ess\,sup}_{E_{4n} \setminus E_{k_f}} \left|(\phi'(\rho))^2 \Gamma(\rho,\rho) + \phi'(\rho) N(\rho)\right|. \tag{6.17}$$

Note that $c_n(f)$ is well-defined by (C3) and (6.8) and depends on f since the essential supremum is taken over $E_{4n} \setminus E_{k_f}$. Since N satisfies (6.5) and (6.6), we obtain the following lemma which is the key lemma of this Chapter.

LEMMA **6.3** *Let V be a relatively compact open set in E, $u \in D(\mathcal{E}^0)_{0,b}$ with supp$(u) \subset \overline{V}$ and $\psi \in D(\mathcal{E}^0)_{loc,b}$. Then $e^{\pm\psi} u \in D(\mathcal{E}^0)_b \subset D(N)_b$ and*

$$\mathcal{E}^0(e^{\psi}u, e^{-\psi}u) + \int_V e^{\psi}u \cdot N(e^{-\psi}u) d\mu \ge \mathcal{E}^0(u,u) - c\int_V u^2 d\mu, \tag{6.18}$$

where

$$c := \operatorname*{ess\,sup}_V |\Gamma(\psi,\psi) + N(\psi)|.$$

Proof $e^{\pm\psi}u \in D(\mathcal{E}^0)_b$ follows since $(e^{\pm\psi}-1)u \in D(\mathcal{E}^0)_b$. Since $(N, D(N))$ satisfies (6.5) and (6.6) and Γ satisfies the Leibniz and Chain rules,

$$\begin{aligned}
\mathcal{E}^0(e^{\psi}u, e^{-\psi}u) + \int_V e^{\psi}u \cdot N(e^{-\psi}u) d\mu &= \mathcal{E}^0(u,u) - \int_V (\Gamma(\psi,\psi) + N(\psi))\, u^2 d\mu \\
&\ge \mathcal{E}^0(u,u) - c\int_V u^2 d\mu.
\end{aligned}$$

□

For $s > 0$, we have by (6.3)

$$\begin{aligned} & \frac{1}{2}\frac{d}{ds}\|\widehat{T}_s^{\psi_n} f\|^2_{L^2(V_n,\mu)} \\ = & \int \widehat{L}^n(\widehat{T}^n_s f)\cdot e^{\psi_n}\widehat{T}_s^{\psi_n} f d\mu \\ = & -\mathcal{E}^0\left(e^{\psi_n}\widehat{T}_s^{\psi_n} f, e^{-\psi_n}\widehat{T}_s^{\psi_n} f\right) - \int_{V_n} e^{\psi_n}\widehat{T}_s^{\psi_n} f\cdot N(e^{-\psi_n}\widehat{T}_s^{\psi_n} f)d\mu. \end{aligned}$$

Replacing u by $\widehat{T}_s^{\psi_n} f$, $s > 0$ and ψ by ψ_n in (6.18), we obtain

$$\frac{1}{2}\frac{d}{ds}\|\widehat{T}_s^{\psi_n} f\|^2_{L^2(V_n,\mu)} \le -\mathcal{E}^0\left(\widehat{T}_s^{\psi_n} f, \widehat{T}_s^{\psi_n} f\right) + c_n(f)\int_{V_n}\left(\widehat{T}_s^{\psi_n} f\right)^2 d\mu.$$

Consequently, $\frac{d}{ds}\|\widehat{T}_s^{\psi_n} f\|^2_{L^2(V_n,\mu)} \le 2c_n(f)\|\widehat{T}_s^{\psi_n} f\|^2_{L^2(V_n,\mu)}$, i.e.

$$\left\|\widehat{T}_s^{\psi_n} f\right\|_{L^2(V_n,\mu)} \le e^{c_n(f)s}\|f\|_{L^2(E,\mu)}.$$

By Fubini and Jensen,

$$\|\widehat{v}_t\|^2_{L^2(V_n,\mu)} \le t\int_0^t\int\left(\widehat{T}_s^{\psi_n} f\right)^2 d\mu ds \le t\int_0^t e^{2c_n(f)s}ds\|f\|^2_{L^2(E,\mu)}$$

Hence, we get

$$\|\widehat{v}_t\|^2_{L^2(V_n,\mu)} \le \frac{t}{2c_n(f)}\left(e^{2c_n(f)t} - 1\right)\|f\|^2_{L^2(E,\mu)} \tag{6.19}$$

and

$$\|\widehat{v}_t\|^2_{L^2(V_n,\mu)} \le t^2 e^{2c_n(f)t}\|f\|^2_{L^2(E,\mu)}. \tag{6.20}$$

Next, using (6.18) again we obtain

$$\begin{aligned} \left(\widehat{v}_t, \widehat{T}_t^{\psi_n} f\right) - (\widehat{v}_t, f) &= \int\int_0^t \frac{d}{du}\widehat{T}^n_u f du\cdot e^{\psi_n}\int_0^t \widehat{T}_s^{\psi_n} f ds d\mu \\ &= \int \widehat{L}^n\left(\int_0^t \widehat{T}^n_u f du\right)\cdot e^{\psi_n}\int_0^t \widehat{T}_s^{\psi_n} f ds d\mu \\ &\qquad - \int_{V_n} e^{\psi_n}\int_0^t \widehat{T}_s^{\psi_n} f ds\cdot N\left(\int_0^t \widehat{T}^n_u f du\right)d\mu \\ &= -\mathcal{E}^0\left(e^{\psi_n}\int_0^t \widehat{T}_s^{\psi_n} f ds, e^{-\psi_n}\int_0^t \widehat{T}_u^{\psi_n} f du\right) \\ &\qquad - \int_{V_n} e^{\psi_n}\int_0^t \widehat{T}_s^{\psi_n} f ds\cdot N\left(e^{-\psi_n}\int_0^t \widehat{T}_u^{\psi_n} f du\right)d\mu \\ &\le -\mathcal{E}^0\left(\widehat{v}_t, \widehat{v}_t\right) + c_n(f)\left\|\widehat{v}_t\right\|^2_{L^2(V_n,\mu)}. \end{aligned}$$

Thus, we get by (6.19) and (6.20)

$$\begin{aligned}
\mathcal{E}^0\left(\widehat{v}_t, \widehat{v}_t\right) &\le c_n(f)\, \|\widehat{v}_t\|^2_{L^2(V_n,\mu)} + (\widehat{v}_t, f) - \left(\widehat{v}_t, \widehat{T}_t^{\psi_n} f\right) \\
&\le c_n(f)\, \|\widehat{v}_t\|^2_{L^2(V_n,\mu)} + \|\widehat{v}_t\|_{L^2(V_n,\mu)} \|f\|_{L^2(E,\mu)} + \|\widehat{v}_t\|_{L^2(V_n,\mu)} \left\|\widehat{T}_t^{\psi_n} f\right\|_{L^2(V_n,\mu)} \\
&\le \frac{t}{2}\left(e^{2c_n(f)t} - 1\right) \|f\|^2_{L^2(E,\mu)} + te^{c_n(f)t}\|f\|^2_{L^2(E,\mu)} + te^{2c_n(f)t}\|f\|^2_{L^2(E,\mu)} \\
&\le 3te^{2c_n(f)t}\|f\|^2_{L^2(E,\mu)}. \qquad (6.21)
\end{aligned}$$

Consequently, using the estimates (6.19), (6.20), (6.21) in (6.14), we get

$$\left|\int_0^t \int \frac{d}{ds}\widehat{T}_s^n f \cdot \chi_n d\mu ds\right| \qquad (6.22)$$

$$\le \frac{e^{\phi(k_f) - \phi(2n) + c_n(f)t}}{2n} \|f\|_{L^2(E,\mu)} \left((\sqrt{3ta_n} + b_n t)\mu(E_{4n} \setminus E_{2n})^{1/2} + t\|N(\rho)\|_{L^2(E_{4n} \setminus E_{2n},\mu)}\right).$$

Let

$$\widehat{A}_n(\phi) := (\sqrt{a_n} + b_n)\mu(E_{4n} \setminus E_{2n})^{1/2} + \|N(\rho)\|_{L^2(E_{4n} \setminus E_{2n},\mu)} \qquad (6.23)$$

where a_n and b_n are defined as in (6.15), (6.16) respectively. Note that $\widehat{A}_n(\phi)$ depends on the choice of ϕ but does not depend on f. Lemma 6.2 now leads to the following theorem.

THEOREM **6.1**

(i) Let f be as in (A) and suppose that there exists a continuously differentiable function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ and $\phi(r) \nearrow +\infty$ as $r \nearrow +\infty$, such that for some constant $T > 0$

$$\limsup_{n\to\infty} \frac{e^{-\phi(2n) + c_n(f)T}}{n} \widehat{A}_n(\phi) = 0 \qquad (6.24)$$

where $\widehat{A}_n(\phi)$ is defined as in (6.23). Then, we obtain

$$\int_E \widehat{T}_t f d\mu = \int_E f d\mu.$$

(ii) Assume that (6.24) holds for at least one triple (f, ϕ, T) as in (i). Then (6.24) holds for the triple (g, ϕ, T), for any $g \in D_0$ (see (6.9) for the definition of D_0). In particular, if additionally D_0 is dense in $L^1(E, \mu)$, then $(T_t)_{t>0}$ is conservative.

Proof (i) is a direct consequence of (6.10), (6.22) and (6.24). We now prove (ii). Let (f, ϕ, T) be as in (i) and $g \in D_0$. It suffices to show that

$$\limsup_{n\to\infty} \frac{e^{-\phi(2n)+c_n(g)T}}{n} \widehat{A}_n(\phi) = 0$$

where

$$c_n(g) = \operatorname*{ess\,sup}_{E_{4n}\smallsetminus E_{k_g}} \left|(\phi'(\rho))^2\Gamma(\rho,\rho) + \phi'(\rho)N(\rho)\right|.$$

If $k_g \geq k_f$, then $E_{k_f} \subset E_{k_g}$ and so $c_n(g) \leq c_n(f)$. Thus (6.24) for (f, ϕ, T) implies (6.24) for (g, ϕ, T). If $k_g < k_f$, then

$$c_n(g) \leq c_n(f) + \operatorname*{ess\,sup}_{E_{k_f}\smallsetminus E_{k_g}} \left|(\phi'(\rho))^2\Gamma(\rho,\rho) + \phi'(\rho)N(\rho)\right| \leq c_n(f) + L$$

for some constant $L \geq 0$, since $\operatorname{ess\,sup}_{E_{k_f}\smallsetminus E_{k_g}} \left|(\phi'(\rho))^2\Gamma(\rho,\rho) + \phi'(\rho)N(\rho)\right|$ is finite by (C3) and (6.8). Thus (6.24) holding for the triple (f, ϕ, T) again implies (6.24) for the triple (g, ϕ, T). If additionally D_0 is dense, then $(T_t)_{t>0}$ is conservative by Lemma 6.2.

□

We formulate the condition of Theorem 6.1(ii) as

(C4) D_0 is dense in $L^1(E, \mu)$.

It is clear that (C4) implies (A). Now, we use Theorem 6.1 to develop the following explicit sufficient conditions for conservativeness of $(T_t)_{t>0}$.

COROLLARY **6.1** *Assume that (C1)-(C4) hold.*

(i) *Suppose there are constants* $M, C > 0$, $0 < \alpha < 1$ *and* $0 \le \beta < 2$, *such that*

$$\left|\Gamma(\rho,\rho) + \frac{(\rho+1)N(\rho)}{C(2-\beta)(\log(\rho+1))^{1-\beta}}\right| \le M(\rho+1)^2(\log(\rho+1))^{\beta} \qquad (6.25)$$

μ-a.e. outside some arbitrary compact subset K *of* E *with* $K \supset K_0$ *and*

$$\widehat{A}_n(\phi) \le n\exp(\alpha C\,(\log(n+1))^{2-\beta}),$$

for $n \gg 1$, *where* $\phi(r) = C(\log(r+1))^{2-\beta}$. *Then* $(T_t)_{t>0}$ *is conservative.*

(ii) *Suppose there are constants* $M, C > 0$ *and* $0 < \alpha < 1$, *such that*

$$\left|\Gamma(\rho,\rho) + \frac{1}{C}(\rho+1)(\log(\rho+1))N(\rho)\right| \le M(\rho+1)^2(\log(\rho+1))^2$$

μ-a.e. outside some arbitrary compact subset K *of* E *with* $K \supset K_0$ *and*

$$\widehat{A}_n(\phi) \le n\log(n+1)^{C\alpha},$$

for $n \gg 1$, *where* $\phi(r) = C\log(\log(r+1)+1)$. *Then* $(T_t)_{t>0}$ *is conservative.*

(iii) *Suppose that there are constants* $M, C > 0$ *and* $0 < \alpha < 2$ *such that*

$$\left|\Gamma(\rho,\rho) + \frac{N(\rho)}{C\rho}\right| \le M$$

μ-a.e. outside some arbitrary compact subset K *of* E *with* $K \supset K_0$ *and*

$$\widehat{A}_n(\phi) \le n\exp(\alpha C n^2)$$

for $n \gg 1$, *where* $\phi(r) = \frac{Cr^2}{2}$. *Then* $(T_t)_{t>0}$ *is conservative.*

Proof (i) Assume there are constants $M, C > 0$, $0 < \alpha < 1$ and $0 \le \beta < 2$ such that (6.25) holds. Let

$$\phi(r) := C(\log(r+1))^{2-\beta}.$$

Since $0 \le \beta < 2$, $\phi(r)$ is increasing in $r > 0$ and

$$\phi'(r) = \frac{C(2-\beta)}{(r+1)}(\log(r+1))^{1-\beta}.$$

By (C4), we can choose $g \in D_0$ with $\mathrm{supp}(g) \neq \emptyset$. By definition of k_g, we know $K_0 \subset K \subset E_{k_g}$. Hence by (6.25), we obtain that

$$c_n(g) \le \operatorname*{ess\,sup}_{E_{4n} \setminus K} \left|(\phi'(\rho))^2\right| \cdot \left|\Gamma(\rho,\rho) + \frac{N(\rho)}{\phi'(\rho)}\right| \le M'(\log(4n+1))^{2-\beta}$$

where $M' > 0$ is some constant depending only on M, C and β. Subsequently, for $n \ge k_g$

$$\begin{aligned} &\frac{\widehat{A}_n(\phi)}{n} \exp(-\phi(2n) + c_n(g)T) \\ \le\ & \exp\left(\alpha C(\log(n+1))^{2-\beta} - C(\log(2n+1))^{2-\beta} + M'T(\log(4n+1))^{2-\beta}\right). \end{aligned}$$

Let $T := \frac{C(1-\alpha)}{2M'} > 0$. Then the right hand side of the above inequality tends to 0 as $n \to \infty$ and so (6.24) of Theorem 6.1(i) holds for the triple (g, ϕ, T). Using (C4), Theorem 6.1(ii) applies, i.e. $(T_t)_{t>0}$ is conservative.

(ii) Let $\beta = 2$. Putting

$$\phi(r) := C\log(\log(r+1)+1),$$

we can proceed as in (i) to show that $(T_t)_{t>0}$ is conservative.

(iii) Let $g \in D_0$ with $\mathrm{supp}(g) \neq \emptyset$. For $n \ge k_g$,

$$c_n(g) \le \operatorname*{ess\,sup}_{E_{4n} \setminus K_0} |(\phi'(\rho)^2| \cdot \left|\Gamma(\rho,\rho) + \frac{N(\rho)}{\phi'(\rho)}\right| \le \operatorname*{ess\,sup}_{E_{4n} \setminus K_0} MC^2\rho^2 = 16MC^2n^2,$$

and so

$$\frac{e^{-\phi(2n)+c_n(g)T}}{n}\widehat{A}_n(\phi) \le \exp(\alpha C n^2 - 2Cn^2 + 16MC^2Tn^2).$$

Let $T := \frac{2-\alpha}{32MC} > 0$, then

$$\lim_{n\to\infty} \frac{e^{-\phi(2n)+c_n(g)T}}{n}\widehat{A}_n(\phi) = 0.$$

Applying Theorem 6.1(ii), we obtain that $(T_t)_{t>0}$ is conservative.

$\square$

Chapter 7 Applications to symmetric and non-symmetric Dirichlet forms

7.1 Symmetric Dirichlet forms

In this Section, we apply Theorem 6.1 to symmetric Dirichlet forms. The results turn out to be comparable with the results of [25, Section 3.1] (cf. Example 7.1 and Remark 7.1 below).

Let $(\mathcal{E},\mathcal{F})$ be a symmetric strongly local regular Dirichlet form on $L^2(E,\mu)$ expressed as

$$\mathcal{E}(f,g) = \int_E \Gamma(f,g)(x)\mu(dx), \text{ for } f,g \in \mathcal{F}. \tag{7.1}$$

Let us fix an arbitrary $x_0 \in E$ and denote $d(x,x_0)$ by $d(x)$ for simplicity. Assume

$$d \in \mathcal{F}_{loc} \tag{7.2}$$

and that

$$E_r := \{x \in E : d(x) < r\} \text{ are relatively compact open sets in } E \text{ for any } r > 0. \tag{7.3}$$

Assume further that there exists a compact subset K_0 of E such that

$$\Gamma(d,d) \in L^\infty_{loc}(K_0^c,\mu). \tag{7.4}$$

As we have seen in Remark 6.1, (C1) and (C2) hold with $p = 2$ and $N \equiv 0$. Furthermore, putting $\rho(x) = d(x)$, (C3) also holds by (7.2), (7.3) and (7.4). Since the semigroups $(T_t^n)_{t>0}$, $n \geq 1$ of the part forms $(\mathcal{E}^n,\mathcal{F}^n)$ on $L^2(V_n,\mu)$ are analytic so that in particular $T_t^n f \in D(L^n)$ for any $f \in L^2(E,\mu)$ and $t > 0$, (C4) also holds (obviously $D_0 = L^\infty(E,\mu) \cap L^2(E,\mu)_0$ is dense in $L^1(E,\mu)$). Thus we can use Theorem 6.1 to determine conservativeness of the symmetric Dirichlet form $(\mathcal{E},\mathcal{F})$. More precisely, we have:

PROPOSITION **7.1**

(i) Assume there are constants $M, N > 0$ and $0 \le \beta \le 2$, such that

$$\Gamma(d,d) \le M(d+1)^2 \left(\log(d+1)\right)^\beta, \tag{7.5}$$

μ-a.e. outside some arbitrary compact subset K of E with $K \supset K_0$ and

$$\mu(E_{4n} \setminus E_{2n}) \le \exp(2N\left(\log(n+1)\right)^{2-\beta}), \ \ \textit{if } 0 \le \beta < 2,$$

or

$$\mu(E_{4n} \setminus E_{2n}) \le \log(n+1)^{2N}, \ \ \textit{if } \beta = 2$$

for $n \gg 1$. Then $(T_t)_{t>0}$ is conservative.

(ii) Assume there are constants $M, N > 0$ such that

$$\Gamma(d,d) \le M$$

μ-a.e. outside some arbitrary compact subset K of E with $K \supset K_0$ and

$$\mu(E_{4n} \setminus E_{2n}) \le \exp(2Nn^2)$$

for $n \gg 1$. Then $(T_t)_{t>0}$ is conservative.

Proof (i) Let $0 \le \beta < 2$ and define for $r > 0$,

$$\phi(r) := C\left(\log(r+1)\right)^{2-\beta}$$

where $C > 0$ will be chosen later. Then, $\phi(r)$ is increasing in $r > 0$ and

$$\phi'(r) = \frac{C(2-\beta)}{(r+1)} \left(\log(r+1)\right)^{1-\beta}.$$

Choose $g \in D_0$ with $\mathrm{supp}(g) \neq \varnothing$. For $n \geq k_g$, we have by (7.5)

$$a_n = \operatorname*{ess\,sup}_{E_{4n} \setminus E_{2n}} \Gamma(d,d) \leq M(4n+1)^2(\log(4n+1))^{\beta},$$

$$b_n = \operatorname*{ess\,sup}_{E_{4n} \setminus E_{2n}} \phi'(d)\Gamma(d,d) \leq MC(2-\beta)(4n+1)\log(4n+1)$$

and

$$c_n(g) \leq \operatorname*{ess\,sup}_{E_{4n} \setminus K} (\phi'(d))^2\Gamma(d,d) \leq MC^2(2-\beta)^2 \left(\log(4n+1)\right)^{2-\beta}.$$

Subsequently,

$$e^{-\phi(2n)+c_n(g)T}\mu(E_{4n} \setminus E_{2n})^{1/2}$$
$$\leq \exp\left(-C(\log(2n+1))^{2-\beta} + N(\log(n+1))^{2-\beta} + TMC^2(2-\beta)^2(\log(4n+1))^{2-\beta}\right).$$

Let $C := 3N$ and $T := \frac{1}{9MN(2-\beta)^2} > 0$, then we obtain

$$\lim_{n\to\infty} \frac{e^{-\phi(2n)+c_n(g)T}}{n}\widehat{A}_n(\phi) = \lim_{n\to\infty} \frac{e^{-\phi(2n)+c_n(g)T}}{n}\mu(E_{4n} \setminus E_{2n})^{1/2}(\sqrt{a_n} + b_n) = 0. \tag{7.6}$$

Consequently, by the same arguments in Corollary 6.1, $(T_t)_{t>0}$ is conservative when $0 \leq \beta < 2$.

Let $\beta = 2$. Define

$$\phi(r) := 3N\log(\log(r+1)+1).$$

Then by similar calculations, we can choose $T > 0$ such that (7.6) holds.

(ii) Choosing $\phi(r) := 3Nr^2$ the proof is similar to the one of (i).

□

EXAMPLE **7.1** *(cf. [25, Section 3.1]) Let* $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ *be a symmetric bilinear form in* $L^2(\mathbb{R}^d, dx)$ *defined by*

$$\mathcal{E}(f,g) := \int_{\mathbb{R}^d} \langle A\nabla f, \nabla g\rangle dx,$$

where $A = (a_{ij}) = (a_{ji}) \in L^1_{loc}(\mathbb{R}^d, dx) \cap L^\infty_{loc}(K_0^c, dx)$*,* $1 \le i,j \le d$ *for some compact subset* K_0 *in* $\mathbb{R}^d$*. Assume that for any compact set* K*, there exists a constant* $\nu_K > 0$ *such that*

$$\nu_K|\xi|^2 \le \langle A(x)\xi, \xi\rangle$$

for all $\xi \in \mathbb{R}^d$*,* μ*-a.e.* $x \in K$*. Then* $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ *is closable and its closure* $(\mathcal{E}, \mathcal{F})$ *satisfies (C1)-(C4) with* $p = 2$*,* $N \equiv 0$ *and* $\rho(x) = |x|$*. Indeed, for each relatively compact open subset* V *of* $\mathbb{R}^d$*, there exists a function* $\chi_V \in C_0^\infty(\mathbb{R}^d)$ *such that* $\chi_V \equiv 1$ *on* V*. Then* $\rho\chi_V \in \mathcal{F}$ *and* $\rho\chi_V = \rho$ *on* V*, hence* $\rho \in \mathcal{F}_{loc}$*. Consequently, by Proposition 7.1(i),* $(T_t)_{t>0}$ *is conservative if there exists a constant* $M > 0$ *such that*

$$\frac{\langle A(x)x, x\rangle}{|x|^2} \le M(|x|+1)^2\log(|x|+1)$$

dx*-a.e. outside some compact subset* K *of* $\mathbb{R}^d$ *containing* K_0*.*

REMARK **7.1** *(cf. [25, Section 3.1]) By Proposition 7.1(ii), we recover the result of [33, Remarks p.185 (3.7)]. More precisely, [33, Theorem 4] was devoted to determine the conservativeness for a symmetric strongly local regular Dirichlet form expressed as in (7.1) in case that the topology induced by the intrinsic metric is equivalent to the original topology on* E *and in case that the intrinsic balls are all relatively compact open in* E *(cf. [33, Assumption (A)]). Then by [33, Lemma 1],* $\rho(\cdot) := d^{int}(\cdot, x_0) \in \mathcal{F}_{loc} \cap C(E)$ *for any* $x_0 \in E$ *where* d^{int} *is the* *intrinsic metric and ρ satisfies*

$$\Gamma(\rho,\rho) \le 1.$$

Applying these assumptions to our situation implies $a_n \le 1$ for any $n \ge 1$. Hence (7.1), (7.2), (7.3) and (7.4) are satisfied and thus by Proposition 7.1(ii), $(T_t)_{t>0}$ is conservative if there exists a constant $N > 0$ such that $\mu(E_{4n} \smallsetminus E_{2n}) \le \exp(2Nn^2)$ for $n \gg 1$.

7.2 Sectorial perturbations of symmetric Dirichlet forms on Euclidean space

In this Section, we apply Theorem 6.1 to non-symmetric Dirichlet forms which are divergence free perturbations of symmetric Dirichlet forms on $\mathbb{R}^d$.

Let $E = \mathbb{R}^d$ and $d\mu = \varphi dx$ where $\varphi \in L^1_{loc}(\mathbb{R}^d, dx)$, $\varphi > 0$ dx-a.e. Consider $A = (a_{ij}) = (a_{ji}) \in L^1_{loc}(\mathbb{R}^d,\mu) \cap L^\infty_{loc}(K_0^c,\mu)$, $1 \le i,j \le d$ for some compact subset K_0 in $\mathbb{R}^d$ and suppose for any compact set $K \subset \mathbb{R}^d$, there exists $\nu_K > 0$ such that

$$\nu_K |\xi|^2 \le \langle A(x)\xi, \xi\rangle \tag{7.7}$$

for all $\xi \in \mathbb{R}^d$, μ-a.e. $x \in K$. We assume that the symmetric bilinear form

$$\mathcal{E}^0(f,g) := \int_{\mathbb{R}^d} \langle A(x)\nabla f(x), \nabla g(x)\rangle \mu(dx),\ \ f,g \in C_0^\infty(\mathbb{R}^d)$$

is closable on $L^2(\mathbb{R}^d,\mu)$. Then its closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a symmetric strongly local regular Dirichlet form. We further assume that $B = (B_1, ..., B_d) \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d, \mu)$ satisfies $|B| \in L^\infty_{loc}(K_0^c, \mu)$ and

$$\int_{\mathbb{R}^d} \langle B(x), \nabla f(x)\rangle \mu(dx) = 0 \tag{7.8}$$

for any $f \in C_0^\infty(\mathbb{R}^d)$ and there exists a constant $C > 0$ which is independent of f and g such that

$$\left|\int_{\mathbb{R}^d}\langle B, \nabla f\rangle g d\mu\right| \leq C\mathcal{E}_1^0(f,f)^{1/2}\mathcal{E}_1^0(g,g)^{1/2}, \tag{7.9}$$

for any $f, g \in C_0^\infty(\mathbb{R}^d)$. Consider the non-symmetric bilinear form

$$\mathcal{E}(f,g) := \int \langle A(x)\nabla f(x), \nabla g(x)\rangle \mu(dx) - \int \langle B(x), \nabla f(x)\rangle g(x)\mu(dx),$$

$f, g \in C_0^\infty(\mathbb{R}^d)$. Then $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d, \mu)$ and by (7.8) and (7.9), the closure $(\mathcal{E}, \mathcal{F})$ is a non-symmetric Dirichlet form in the sense of [18, I. Definition 4.5]. By (7.7), (7.8) and (7.9), we obtain

$$\int_{\mathbb{R}^d}\langle B, \nabla v\rangle d\mu = 0, \text{ for any } v \in \mathcal{F}_b.$$

Let $V_n = \{z : |z| < 4n\}$. As in Remark 6.1, we may define the part Dirichlet forms $(\mathcal{E}^n, \mathcal{F}^n)$ corresponding to the increasing sequence of relatively compact open sets $(V_n)_{n\geq 1}$ where $\mathcal{F}^n = \{u \in \mathcal{F} : \widetilde{u} = 0 \text{ q.e. on } V_n^c\}$ (see [23, Section 3.5]). Denote the coform of $(\mathcal{E}^n, \mathcal{F}^n)$ by $(\widehat{\mathcal{E}}^n, \mathcal{F}^n)$ and the associated semigroups of $(\widehat{\mathcal{E}}^n, \mathcal{F}^n)$ by $(\widehat{T}_t^n)_{t>0}$ and the associated linear operators by $(\widehat{L}^n, D(\widehat{L}^n))$ on $L^2(V_n, \mu)$. Then the coform $(\widehat{\mathcal{E}}^n, \mathcal{F}^n)$ is also a non-symmetric Dirichlet form in $L^2(V_n, \mu)$ and

$$\widehat{T}_t f = \lim_{n\to\infty} \widehat{T}_t^n f \ \mu\text{-a.e.}$$

for any $f \in L^2(\mathbb{R}^d, \mu)$ where $\widehat{T}_t^n f := \widehat{T}_t^n(f \cdot 1_{V_n})$. In particular, if $f \geq 0$ μ-a.e., then $\widehat{T}_t^n f \nearrow \widehat{T}_t f$ μ-a.e. as $n \to \infty$. $(\mathcal{E}, \mathcal{F})$ satisfies (C1) with $p = 2$. Furthermore, for $v \in D(\widehat{L}^n)_b$,

$$(-\widehat{L}^n v, u) = \mathcal{E}(u, v) = \mathcal{E}^0(u, v) + \int_{\mathbb{R}^d}\langle B, \nabla v\rangle u d\mu \quad \text{for any } u \in \mathcal{F}_b.$$

Putting $D(N) = \mathcal{F}_{loc,b}$ and $Nv = \langle B, \nabla v\rangle$ imply that (6.3) and (C2) hold. Choose $\rho(x) := |x|$. Then in the same way as in Example 7.1, we find that $\rho \in \mathcal{F}_{loc}$ and by the assumptions on A and B, we obtain

$$\langle A\nabla\rho, \nabla\rho\rangle,\ \langle B, \nabla\rho\rangle \in L^\infty_{loc}(K_0^c, \mu)$$

hence, (C3) holds. By [18, I. Corollary 2.21], $(\widehat{T}_t^n)_{t>0}$ is analytic on $L^2(V_n, \mu)$, hence (C4) holds (i.e. $D_0 = L^\infty(\mathbb{R}^d, \mu) \cap L^2(\mathbb{R}^d, \mu)_0$). Thus we can apply Theorem 6.1 to determine conservativeness.

7.2.1 Example

As an application, consider the non-symmetric Dirichlet form introduced in [29, Section 5]. There φ is a Muckenhoupt $\mathcal{A}_\beta$-weight, $1 \le \beta \le 2$ with $\varphi = \xi^2$, $\xi \in H^{1,2}_{loc}(\mathbb{R}^d, dx)$, $\varphi > 0$ dx-a.e. and

$$\frac{|\nabla\varphi|}{\varphi} \in L^p_{loc}(\mathbb{R}^d, dx)$$

where $p = (d+\varepsilon) \vee 2$ for some $\varepsilon > 0$, $H^{1,2}(\mathbb{R}^d, dx)$ is the usual Sobolev space of order one in $L^2(\mathbb{R}^d, dx)$ and $H^{1,2}_{loc}(\mathbb{R}^d, dx) := \{f : f\cdot\chi \in H^{1,2}(\mathbb{R}^d, dx)$ for any $\chi \in C_0^\infty(\mathbb{R}^d)\}$. Thus the symmetric bilinear form

$$\mathcal{E}^0(f,g) = \int_{\mathbb{R}^d} \langle\nabla f(x), \nabla g(x)\rangle \mu(dx),\ f, g \in C_0^\infty(\mathbb{R}^d)$$

is closable on $L^2(\mathbb{R}^d, \mu)$. Moreover, in [29, Section 5] it is assumed that $B = (B_1, ..., B_d) \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d, \mu)$ is μ-divergence free and

$$|B| \in L^N_{loc}(\mathbb{R}^d, \mu) \cap L^\infty(K_0^c, \mu)$$

for some compact set K_0 and some constant $N \geq \beta d + \log_2 A$, where the constant A is the $\mathcal{A}_\beta$ constant of φ. Then by [29, Section 5], (7.9) holds. The corresponding closure $(\mathcal{E}, \mathcal{F})$ satisfies (C1)-(C4) with $D(N) = \mathcal{F}_{loc,b}$, $Nv = \langle B, \nabla v\rangle$ and $\rho(x) = |x|$ as in Example 7.1 and $D_0 = L^\infty(\mathbb{R}^d, \mu) \cap L^2(\mathbb{R}^d, \mu)_0$. In this situation, $\Gamma(\rho, \rho) = 1$ and

$$\left|\frac{\langle B, \nabla\rho\rangle}{\rho}\right| \leq \|B\|_{L^\infty(K_0^c, \mu)}$$

μ-a.e. on K^c where K is an arbitrary compact subset of $\mathbb{R}^d$ containing K_0 and $\{x \in \mathbb{R}^d : |x| \leq 1\}$. Furthermore, since $\varphi \in \mathcal{A}_\beta$, we get by [43, Proposition 1.2.7] that there exists a constant $N > 0$ such that

$$\mu(E_{4n}) \leq N n^{\beta d}.$$

Thus, for $\phi(r) := \frac{r^2}{2}$ we obtain (cf. (6.23)) for $n \gg 1$

$$\widehat{A}_n(\phi) \leq N\left(1 + 4n + \|B\|_{L^\infty(K_0^c, \mu)}\right) n^{\beta d}.$$

$(T_t)_{t>0}$ is conservative by Corollary 6.1(iii) and we recover the result of [29, Lemma 5.4].

7.3 Sectorial perturbations of sectorial Dirichlet forms

In this Section, we show that Theorem 6.1 is also applicable to non-symmetric Dirichlet forms with non-symmetric diffusion matrix. The key observation is that the anti-symmetric part of the diffusion matrix becomes a μ-divergence free vector field after integration by parts.

Let $E = \mathbb{R}^d$ and $d\mu = \varphi^2 dx$, $\varphi \in H^{1,2}_{loc}(\mathbb{R}^d, dx)$, $\varphi > 0$ dx-a.e. Let $H^{1,2}(\mathbb{R}^d, \mu)$ be the closure of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, \mu)$ with respect to the norm $\left(\int_{\mathbb{R}^d}(|\nabla f|^2 + f^2) d\mu\right)^{1/2}$

and

$$H^{1,2}_{loc}(\mathbb{R}^d,\mu) := \left\{ f : f\cdot\chi \in H^{1,2}(\mathbb{R}^d,\mu) \text{ for any } \chi \in C_0^\infty(\mathbb{R}^d) \right\}.$$

Consider $A = (a_{ij}) \in L^1_{loc}(\mathbb{R}^d,\mu)$, $1 \le i,j \le d$ with symmetric part $\widetilde{A} = (\widetilde{a}_{ij})$, where $\widetilde{a}_{ij} := \frac{1}{2}(a_{ij}+a_{ji}) \in L^\infty_{loc}(K_0^c,\mu)$ for some compact subset K_0 in $\mathbb{R}^d$ and anti-symmetric part $\check{A} = (\check{a}_{ij})$, where $\check{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji}) \in H^{1,2}_{loc}(\mathbb{R}^d,\mu) \cap L^\infty_{loc}(\mathbb{R}^d,\mu)$. Suppose for any compact set $K \subset \mathbb{R}^d$, there exist $\nu_K > 0$ and $L > 0$, such that

$$\max_{1\le i,j\le d} \operatorname*{ess\,sup}_K |\check{a}_{ij}| \le L\cdot\nu_K \quad \text{and} \quad \nu_K|\xi|^2 \le \langle \widetilde{A}(x)\xi,\xi\rangle$$

for all $\xi \in \mathbb{R}^d$, μ-a.e. $x \in K$. Assume that $B = (B_1,...,B_d) \in L^2_{loc}(\mathbb{R}^d,\mathbb{R}^d,\mu)$ satisfies

$$\int_{\mathbb{R}^d} \langle B,\nabla f\rangle d\mu = 0, \tag{7.10}$$

for any $f \in C_0^\infty(\mathbb{R}^d)$. Assume further that there exists a constant $C > 0$ such that

$$\left| \int_{\mathbb{R}^d} \langle B,\nabla f\rangle d\mu \right| \le C\mathcal{E}_1^{\widetilde{A}}(f,f)^{1/2}\mathcal{E}_1^{\widetilde{A}}(g,g)^{1/2}$$

for any $f,g \in C_0^\infty(\mathbb{R}^d)$, where $\mathcal{E}^{\widetilde{A}}(f,g) := \int_{\mathbb{R}^d}\langle \widetilde{A}\nabla f,\nabla g\rangle d\mu$. Likewise, define $\mathcal{E}^{\check{A}}(f,g)$ and $\mathcal{E}^{A}(f,g)$. Set

$$\mathcal{E}^{A,B}(f,g) := \mathcal{E}^A(f,g) - \int_{\mathbb{R}^d}\langle B,\nabla f\rangle g d\mu = \int_{\mathbb{R}^d}\langle A\nabla f,\nabla g\rangle d\mu - \int_{\mathbb{R}^d}\langle B,\nabla f\rangle g d\mu,$$

for any $f,g \in C_0^\infty(\mathbb{R}^d)$. Then $(\mathcal{E}^{A,B}, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d,\mu)$ and its closure $(\mathcal{E}^{A,B},\mathcal{F})$ is a non-symmetric sectorial regular Dirichlet form. Let $(T_t)_{t>0}$ (resp. $(\widehat{T}_t)_{t>0}$) be the C_0-semigroup of contractions on $L^2(\mathbb{R}^d,\mu)$ associated with $(\mathcal{E}^{A,B},\mathcal{F})$, and $(L,D(L))$ (resp. $(\widehat{L},D(\widehat{L}))$ be the corresponding linear operator (resp. co-operator). For $f, g \in C_0^\infty(\mathbb{R}^d)$, we obtain by integration by parts

$$\begin{aligned}
&\mathcal{E}^{\check{A}}(f,g) - \int_{\mathbb{R}^d} \langle B, \nabla f\rangle g d\mu \\
= &-\sum_{i,j}^{d} \int_{\mathbb{R}^d} \left[\check{a}_{ij}\partial_i\partial_j f + \left(\partial_j \check{a}_{ij} + \check{a}_{ij}\frac{2\partial_j\varphi}{\varphi} + B_i\right)\partial_i f\right] g\varphi^2 dx \\
= &-\sum_{i=1}^{d} \int_{\mathbb{R}^d} \underbrace{\left[\sum_{j=1}^{d}\left(\partial_j \check{a}_{ij} + \check{a}_{ij}\frac{2\partial_j\varphi}{\varphi} + B_i\right)\right]}_{=:\beta_i} \partial_i f g\varphi^2 dx \qquad (7.11)
\end{aligned}$$

where $\beta = (\beta_1, ..., \beta_d) \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d, \mu)$ is again a μ-divergence free vector field. Indeed, for $f \in C_0^\infty(\mathbb{R}^d)$, we get by (7.10)

$$\int_{\mathbb{R}^d} \langle \beta, \nabla f\rangle d\mu = \sum_{i=1}^{d} \int_{\mathbb{R}^d} \left[\sum_{j=1}^{d}\left(\partial_j \check{a}_{ij} + \check{a}_{ij}\frac{2\partial_j\varphi}{\varphi}\right)\right]\partial_i f \varphi^2 dx = -\int_{\mathbb{R}^d} \sum_{i,j}^{d} \check{a}_{ij}\partial_i\partial_j f \varphi^2 dx = 0.$$

Moreover, by (7.11) and since $\mathcal{E}^{\check{A}}$ satisfies the strong sector condition, there is a constant $C > 0$ such that

$$\left|\int_{\mathbb{R}^d} \langle \beta - B, \nabla f\rangle g d\mu\right| = \left|\mathcal{E}^{\check{A}}(f,g)\right| \le C\mathcal{E}_1^{\widetilde{A}}(f,f)^{1/2}\mathcal{E}_1^{\widetilde{A}}(g,g)^{1/2} \text{ for any } f,g \in C_0^\infty(\mathbb{R}^d),$$

hence $\left|\int_{\mathbb{R}^d} \langle \beta, \nabla f\rangle g d\mu\right| \le C\mathcal{E}_1^{\widetilde{A}}(f,f)^{1/2}\mathcal{E}_1^{\widetilde{A}}(g,g)^{1/2}$ for some constant $C > 0$. It follows that B and β satisfy the same assumptions and that

$$\mathcal{E}^{A,B}(f,g) = \mathcal{E}^{\widetilde{A}}(f,g) - \int_{\mathbb{R}^d} \langle \beta, \nabla f\rangle g d\mu =: \mathcal{E}^{\widetilde{A},\beta}(f,g)$$

for any $f, g \in C_0^\infty(\mathbb{R}^d)$. Therefore, the closures of $(\mathcal{E}^{A,B}, C_0^\infty(\mathbb{R}^d))$ and $(\mathcal{E}^{\widetilde{A},\beta}, C_0^\infty(\mathbb{R}^d))$ are identical and define the same Dirichlet form. We now assume that $|\beta| \in L^\infty_{loc}(K_0^c, \mu)$.

Let $V_n = E_{4n} = \{z : |z| < 4n\}$, $n \ge 1$. Then $(V_n)_{n\ge1}$ is a sequence of relatively compact open sets. As in Section 7.2, let $(\mathcal{E}^n, \mathcal{F}^n)$ be the part Dirichlet forms on $L^2(V_n, \mu)$ of $(\mathcal{E}^{\widetilde{A},\beta}, \mathcal{F})$ (see [23, Section 3.5]). Let $(\widehat{\mathcal{E}}^n, \mathcal{F}^n)$ be the coform of

$(\mathcal{E}^n, \mathcal{F}^n)$, $(\widehat{T}_t^n)_{t>0}$ be the associated semigroups of $(\widehat{\mathcal{E}}^n, \mathcal{F}^n)$ and $(\widehat{L}^n, D(\widehat{L}^n))$ be the associated linear operators on $L^2(V_n, \mu)$. Then,

$$\widehat{T}_t f = \lim_{n\to\infty} \widehat{T}_t^n f \ \mu\text{-a.e.}$$

for any $f \in L^2(\mathbb{R}^d, \mu)$ where $\widehat{T}_t^n f := \widehat{T}_t^n(f \cdot 1_{V_n})$. In particular, if f is non-negative, then $\widehat{T}_t^n f \nearrow \widehat{T}_t f$ μ-a.e. as $n \to \infty$. $(\mathcal{E}^{\widetilde{A},\beta}, \mathcal{F})$ satisfies (C1) with $p = 2$. Furthermore, for $v \in D(\widehat{L}^n)_b$,

$$(-\widehat{L}^n v, u) = \mathcal{E}^{\widetilde{A},-\beta}(u,v) = \mathcal{E}^{\widetilde{A}}(u,v) + \int_{\mathbb{R}^d} \langle \beta, \nabla v\rangle u d\mu, \quad \text{for any } u \in \mathcal{F}_b.$$

Putting $D(N) = \mathcal{F}_{loc,b}$ and $Nv = \langle \beta, \nabla v\rangle$ imply that (6.3) and (C2) hold with $(\mathcal{E}^0, D(\mathcal{E}^0)) = (\mathcal{E}^{\widetilde{A}}, \mathcal{F})$. Let $\rho(x) := |x|$ then $\rho \in \mathcal{F}_{loc}$ as in Example 7.1. We further obtain by the assumptions on $\widetilde{A}$ and β, that

$$\langle \widetilde{A}\nabla\rho, \nabla\rho\rangle,\ \langle \beta, \nabla\rho\rangle \in L^\infty_{loc}(K_0^c, \mu).$$

Hence (C3) holds. Since $(\widehat{\mathcal{E}}^n, \mathcal{F}^n)$ satisfies the weak sector condition for each $n \geq 1$, $(\widehat{T}_t^n)_{t>0}$ are analytic, i.e. (C4) holds. Consequently, by Corollary 6.1(i) with $\phi(r) := C\log(r+1)$, $\rho(x) = |x|$, if there are constants $M, C > 0$, and $0 < \alpha < 1$ such that

$$\left| \frac{\langle \widetilde{A}(x)x, x\rangle}{|x|^2} + \frac{(|x|+1)}{C|x|}\langle \beta(x), x\rangle \right| \leq M(|x|+1)^2 \log(|x|+1), \tag{7.12}$$

dx-a.e. outside some compact subset K of $\mathbb{R}^d$ with $K \supset K_0$ and

$$\widehat{A}_n(\phi) \leq n(n+1)^{\alpha C} \tag{7.13}$$

for $n \gg 1$, then $(T_t)_{t>0}$ is conservative.

7.3.1 Example

The sufficient criteria (7.12) and (7.13) for conservativeness extend the result of [40] in the sense that we can also consider invariant measures $\mu = \varphi^2 dx$ where $\varphi \not\equiv 1$. In this example, we show that we can also recover the result of [40] to some extend in case $\varphi \equiv 1$.

Let $d \geq 3$ and $\varphi^2 \equiv 1$, i.e. μ is the Lebesgue measure. Assume further that for $B = (B_1, ..., B_d) \in L^d_{loc}(\mathbb{R}^d, \mathbb{R}^d, dx)$, there exist constants $L_i > 0$, such that for $1 \leq i \leq d$

$$\min\{\|B_i^2\|_{L^\infty(E_n)}, \|B_i\|_{L^d(E_n)}\} \leq L_i \nu_{\overline{E}_n}.$$

Then by [40, Section 2], $(\mathcal{E}^{A,B}, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d, dx)$ and the closure $(\mathcal{E}^{A,B}, \mathcal{F})$ satisfies the weak sector condition. Thus, we are able to apply (7.12) and (7.13) to $(\mathcal{E}^{A,B}, \mathcal{F})$ in order to determine the conservativeness. For instance, if there exists a constant $M_0 > 1$ such that

$$\frac{\langle \widetilde{A}(x)x, x\rangle}{|x|^2} + |\langle \beta(x), x\rangle| \leq M_0(|x|+1)^2 \log(|x|+1) \tag{7.14}$$

μ-a.e. outside some compact subset K of $\mathbb{R}^d$ with $K \supset K_0 \cup \{x : |x| \leq 1\}$, then $(\mathcal{E}^{A,B}, \mathcal{F})$ is conservative. Indeed, by (7.14)

$$\left|\frac{\langle \widetilde{A}(x)x, x\rangle}{|x|^2} + \frac{(|x|+1)}{C|x|}\langle \beta(x), x\rangle\right| \leq M_0\left(1 + \frac{2}{C}\right)(|x|+1)^2 \log(|x|+1) \tag{7.15}$$

and

$$\left|\frac{\langle \beta(x), x\rangle}{|x|}\right| \leq M_0(|x|+1)^2 \tag{7.16}$$

μ-a.e. on K^c. Let $\phi(r) := C\log(r+1)$ where the constant $C > 0$ will be chosen later. It follows from (7.14), (7.15) and (7.16) that

$$\widehat{A}_n(\phi) = (\sqrt{a_n} + b_n)\mu(E_{4n} \setminus E_{2n})^{1/2} + \left(\int_{E_{4n}\setminus E_{2n}} \left|\frac{\langle \beta(x), x\rangle}{|x|}\right|^2 dx\right)^{1/2} \leq M' n^{d/2+2}$$

for some constant $M' > 0$. Consequently, putting $C = \frac{d}{2}+3$, $M = M_0\left(1+\frac{2}{C}\right)$ and $\alpha = \frac{C-1}{C}$ implies there are constants $M, C > 0$, and $0 < \alpha < 1$ such that (7.12) and (7.13) hold and $(T_t)_{t>0}$ is conservative.

REMARK **7.2** *Compared with the estimate [40, p. 422], (7.14) is a slightly stronger condition. Our aim was to demonstrate how quickly Corollary 6.1 can lead to acceptable results. Later, by applying Corollary 6.1 more consciously we will see that* $\left|\frac{\langle \widetilde{A}(x)x,x\rangle}{|x|^2}\right|$ *in (7.12) is allowed to have a cubic growth if* $\langle \beta(x), x\rangle$ *can compensate it (see Subsection 8.2.2 below).*

Chapter 8 Non-sectorial applications on Euclidean space

In this Chapter, we consider non-sectorial perturbations of symmetric Dirichlet forms on Euclidean space as introduced in Chapter 4. For the convenience of the reader, we explain in concise form the construction of the underlying generalized Dirichlet form $\mathcal{E}$ from [10], how the constructed generalized Dirichlet form fits into the frame of Chapter 6, as well as some of its main properties. Subsequently, we apply the conservativeness criterion of Chapter 6 to the concrete situation and present explicit examples.

8.1 The construction scheme

Let $E \subset \mathbb{R}^d$ be either open or closed. Let E and φ be as in Chapter 4. Consider $A = (a_{ij}) = (a_{ji}) \in L^1_{loc}(E,\mu)$, $1 \le i,j \le d$ and suppose for each relatively compact open set $V \subset E$, there exists $\nu_V > 0$ such that

$$\nu_V^{-1}|\xi|^2 \le \langle A(x)\xi,\xi\rangle \le \nu_V|\xi|^2 \tag{8.1}$$

for all $\xi \in \mathbb{R}^d$, μ-a.e. $x \in V$. We assume that

$$\mathcal{E}^0(f,g) := \int_E \langle A(x)\nabla f(x), \nabla g(x)\rangle \mu(dx),\ \ f,g \in C_0^\infty(E)$$

is closable on $L^2(E,\mu)$. Denote the closure of $(\mathcal{E}^0, C_0^\infty(E))$ on $L^2(E,\mu)$ by $(\mathcal{E}^0, D(\mathcal{E}^0))$. Then $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a symmetric regular Dirichlet form on $L^2(E,\mu)$. Let $(L^0, D(L^0))$ be the linear operator corresponding to $(\mathcal{E}^0, D(\mathcal{E}^0))$ on $L^2(E,\mu)$

and $(T_t^0)_{t>0}$ be the C_0-semigroup corresponding to $(L^0, D(L^0))$.
Let $B := (B_1, \dots, B_d) \in L^2_{loc}(E, \mathbb{R}^d, \mu)$ satisfy

$$\int_E \langle B(x), \nabla f(x) \rangle \mu(dx) = 0$$

for any $f \in C_0^\infty(E)$.
The following construction from Chapter 4 works for any increasing sequence of relatively compact open sets $(V_n)_{n\geq 1}$ in E such that $\overline{V}_n \subset V_{n+1}$, $n \geq 1$, and $\bigcup_{n\geq 1} V_n = E$. Since we need to assume (C3) later and want to simplify notations we assume from now on that

(B) there exists a non-negative continuous function $\rho \in D(\mathcal{E}^0)_{loc}$ such that

$$E_n := \{x \in E : \rho(x) < n\}$$

is a relatively compact open set in E and $\bigcup_{n\geq 1} E_n = E$ and $\langle B, \nabla \rho \rangle \in L^\infty_{loc}(K_0^c, \mu)$ for some compact subset K_0 in E.

Let

$$V_n := E_{4n}, \;\; n \geq 1.$$

Then $(V_n)_{n\geq 1}$ is an increasing sequence of relatively compact open sets in E such that $\overline{V}_n \subset V_{n+1}$ and $\bigcup_{n\geq 1} V_n = E$. Let $C_0^\infty(V_n) := \{u \in C_0^\infty(E) : \mathrm{supp}(u) \subset V_n\}$ and $(\mathcal{E}^{0,n}, D(\mathcal{E}^{0,n}))$ be the symmetric Dirichlet form on $L^2(V_n, \mu)$ given as the closure of

$$\mathcal{E}^{0,n}(f, g) := \int_{V_n} \langle A(x) \nabla f(x), \nabla g(x) \rangle \mu(dx), \;\; f, g \in C_0^\infty(V_n).$$

Let $(L^{0,n}, D(L^{0,n}))$ be the closed linear operator on $L^2(V_n, \mu)$ associated with $(\mathcal{E}^{0,n}, D(\mathcal{E}^{0,n}))$. Then, by (i) and (ii) in Chapter 5 (cf. also [36, Proposition

1.1]), there exists a closed linear operator $(\overline{L}^n, D(\overline{L}^n))$ on $L^1(V_n, \mu)$ which is the closure of

$$L^{0,n}u + \langle B, \nabla u\rangle, u \in D(L^{0,n})_b$$

on $L^1(V_n, \mu)$ and which generates a sub-Markovian C_0-semigroup of contractions on $L^1(V_n, \mu)$. Let $(L^n, D(L^n))$ be the part of $(\overline{L}^n, D(\overline{L}^n))$ on $L^2(V_n, \mu)$ and $(T_t^n)_{t>0}$ be its sub-Markovian C_0-semigroup on $L^2(V_n, \mu)$. Proceeding in the same way as just explained, there exists a linear operator $(\widehat{L}^n, D(\widehat{L}^n))$ on $L^1(V_n, \mu)$ which is the closure of

$$L^{0,n}v - \langle B, \nabla v\rangle,\ v \in D(L^{0,n})_b$$

on $L^1(V_n, \mu)$ and which satisfies $D(\widehat{L}^n)_b \subset D(\mathcal{E}^{0,n})_b$

$$-\int_{V_n} \widehat{L}^n v v d\mu = \mathcal{E}^{0,n}(v, v)$$

and

$$-\int_{V_n} \widehat{L}^n v u d\mu = \mathcal{E}^0(u, v) + \int_{V_n} \langle B, \nabla v\rangle u d\mu \tag{8.2}$$

for any $v \in D(\widehat{L}^n)_b$ and $u \in D(\mathcal{E}^{0,n})_b$. Let $(\widehat{T}_t^n)_{t>0}$ be the C_0-semigroup of contractions on $L^1(V_n, \mu)$ corresponding to $(\widehat{L}^n, D(\widehat{L}^n))$. Let $(G_\alpha^n)_{\alpha>0}$ (resp. $(\widehat{G}_\alpha^n)_{\alpha>0}$) be the resolvent of $(T_t^n)_{t>0}$ (resp. $(\widehat{T}_t^n)_{t>0}$) on $L^2(V_n, \mu)$ (resp. $L^1(V_n, \mu)$). Define for $f \in L^2(E, \mu)$,

$$G_\alpha^n f := G_\alpha^n(f \cdot 1_{V_n}),\ \alpha > 0.$$

Then $(G_\alpha^n)_{\alpha>0}$, $n \geq 1$, gives rise to a sub-Markovian C_0-resolvent of contractions on $L^2(E, \mu)$. Indeed, let $f \in L^2(E, \mu)_b$, with $f \geq 0$ μ-a.e. and $\alpha > 0$. Let $w_\alpha := G_\alpha^n f - G_\alpha^{n+1} f$. Then $w_\alpha \in D(\mathcal{E}^{0,n+1})_b$ and $w_\alpha^+ \in D(\mathcal{E}^{0,n})_b$. Since

$$\mathcal{E}^0(w_\alpha^+, w_\alpha^-) = \mathcal{E}^0(w_\alpha^+, w_\alpha^+ - w_\alpha) = -\mathcal{E}^0((-w_\alpha) \wedge 0, (-w_\alpha) - (-w_\alpha) \wedge 0) \leq 0$$

and

$$\int_{V_{n+1}} \langle B, \nabla w_\alpha \rangle w_\alpha^+ d\mu = \int_{V_{n+1}} \langle B, \nabla w_\alpha^+ \rangle w_\alpha^+ d\mu = 0,$$

we obtain

$$\begin{aligned} \mathcal{E}_\alpha^0(w_\alpha^+, w_\alpha^+) &\leq \mathcal{E}_\alpha^0(w_\alpha, w_\alpha^+) - \int_{V_{n+1}} \langle B, \nabla w_\alpha \rangle w_\alpha^+ d\mu \\ &= \int_{V_{n+1}} (\alpha - L^n) G_\alpha^n f w_\alpha^+ d\mu - \int_{V_{n+1}} (\alpha - L^{n+1}) G_\alpha^{n+1} f w_\alpha^+ d\mu = 0. \end{aligned}$$

Thus, $w_\alpha^+ = 0$ μ-a.e., i.e. $G_\alpha^n f \leq G_\alpha^{n+1} f$ μ-a.e. (cf. [36, Lemma 1.6]). Define for $f \in L^2(E,\mu)_b$, with $f \geq 0$ μ-a.e.

$$G_\alpha f \coloneqq \lim_{n\to\infty} G_\alpha^n f.$$

Let $f \in L^2(E,\mu)$, $f \geq 0$ and $(f_n)_{n\geq 1} \subset L^2(E,\mu)_b$ with $0 \leq f_n \leq f_{n+1}$ μ-a.e. for every $n \geq 1$ be such that $f_n \to f$ in $L^2(E,\mu)$ as $n \to \infty$. Then

$$G_\alpha f \coloneqq \lim_{n\to\infty} G_\alpha f_n$$

exists μ-a.e. since it is an increasing sequence. For general $f \in L^2(E,\mu)$, let $G_\alpha f \coloneqq G_\alpha f^+ - G_\alpha f^-$. By [10], one can see $(G_\alpha)_{\alpha>0}$ is a sub-Markovian C_0-resolvent of contractions on $L^2(E,\mu)$ provided

(C) $D(L^0)_{0,b}$ is a dense subset of $L^1(E,\mu)$,

which we assume from now on. Let $(L, D(L))$ be the generator of $(G_\alpha)_{\alpha>0}$ and $(T_t)_{t>0}$ be the C_0-semigroup associated with $(L, D(L))$. Let $(\widehat{L}, D(\widehat{L}))$ be the adjoint operator of $(L, D(L))$ and $(\widehat{T}_t)_{t>0}$ (resp. $(\widehat{G}_\alpha)_{\alpha>0}$) be the C_0-semigroup (resp. C_0-resolvent) associated with $(\widehat{L}, D(\widehat{L}))$. Then, we obtain a generalized Dirichlet form $\mathcal{E}$ defined by

$$\mathcal{E}(u,v) \coloneqq \begin{cases} (-Lu, v) & u \in D(L),\ v \in L^2(E,\mu) \\ (-\widehat{L}v, u) & u \in L^2(E,\mu),\ v \in D(\widehat{L}), \end{cases}$$

satisfying $D(L) \subset D(\mathcal{E}^0)$,

$$\mathcal{E}(u,v) = \mathcal{E}^0(u,v) - \int_E \langle B, \nabla u\rangle v d\mu,\ u \in D(L)_b, v \in D(\mathcal{E}^{0,n})_b$$

for some $n \geq 1$ and

$$\mathcal{E}^0(u,u) \leq \mathcal{E}(u,u),\ u \in D(L),$$

i.e. $\mathcal{A} \equiv 0$ on $\mathcal{V} = L^2(E,\mu)$ in the beginning of Chapter 2.

8.2 Conservativeness

By the construction of $(\widehat{T}_t)_{t>0}$,

$$\widehat{T}_t f = \lim_{n\to\infty} \widehat{T}_t^n (f \cdot 1_{V_n})$$

holds for any $f \in L^1(E,\mu) \cap L^\infty(E,\mu)$. Thus, (C1) holds with $p = 1$. Let $D(N) = D(\mathcal{E}^0)_{loc,b}$ and

$$Nv := \langle B, \nabla v\rangle$$

then by (8.2), (C2) holds. By assumption (B), (C3) holds. By the construction of $(\widehat{L}^n, D(\widehat{L}^n))$,

$$f \in D(\widehat{L}^n) \text{ whenever } f \in D(L^0)_{0,b} \text{ for } n \geq k_f,$$

i.e. (C) implies that (A) holds. Since

$$D(L^0)_{0,b} \subset D_0 = \{f : f \in L^\infty(E,\mu) \cap L^2(E,\mu)_0 \ \text{ such that } \\ \widehat{T}_s^n f \in D(\widehat{L}^n),\ \text{for any } n \geq k_f,\ s \in [0,t]\},$$

(C) also implies (C4). Thus, under the assumptions (B) and (C), Corollary 6.1 applies with $\Gamma(\rho,\rho) = \langle A\nabla\rho, \nabla\rho\rangle$, $N(\rho) = \langle B, \nabla\rho\rangle$, ρ as in (B). This gives the following corollary. Recall that in the present situation

$$\widehat{A}_n(\phi) = \left(\sqrt{\operatorname*{ess\,sup}_{E_{4n}\setminus E_{2n}} \langle A\nabla\rho,\nabla\rho\rangle} + \operatorname*{ess\,sup}_{E_{4n}\setminus E_{2n}} \phi'(\rho)\langle A\nabla\rho,\nabla\rho\rangle\right)\mu\left(E_{4n}\setminus E_{2n}\right)^{1/2}$$
$$+\|\langle B,\nabla\rho\rangle\|_{L^2(E_{4n}\setminus E_{2n},\mu)}.$$

COROLLARY **8.1** *Assume (B) and (C) and the basic assumptions on φ, A, B of Section 8.1.*

(i) Assume there are constants $M, C>0$, $0<\alpha<1$ and $0\le\beta<2$ such that

$$\left|\langle A\nabla\rho,\nabla\rho\rangle + \frac{(\rho+1)\langle B,\nabla\rho\rangle}{C(2-\beta)(\log(\rho+1))^{1-\beta}}\right| \le M(\rho+1)^2(\log(\rho+1))^{\beta},$$

μ-a.e. outside some arbitrary compact subset K of E with $K\supset K_0$ and

$$\widehat{A}_n(\phi)\le n\exp(\alpha C\,(\log(n+1))^{2-\beta}),\ \ \phi(r)=C(\log(r+1))^{2-\beta},$$

for $n\gg 1$. Then $(T_t)_{t>0}$ is conservative.

(ii) Assume there are constants $M, C>0$ and $0<\alpha<1$

$$\left|\langle A\nabla\rho,\nabla\rho\rangle + \frac{1}{C}(\rho+1)(\log(\rho+1))\langle B,\nabla\rho\rangle\right| \le M(\rho+1)^2(\log(\rho+1))^2,$$

μ-a.e. outside some arbitrary compact subset K of E with $K\supset K_0$ and

$$\widehat{A}_n(\phi)\le n\log(n+1)^{C\alpha},\ \ \phi(r)=C\log(\log(r+1)+1)$$

for $n\gg 1$. Then $(T_t)_{t>0}$ is conservative.

(iii) Assume that there are constants $M, C>0$ and $0<\alpha<2$ such that

$$\left|\langle A\nabla\rho,\nabla\rho\rangle + \frac{\langle B,\nabla\rho\rangle}{C\rho}\right| \le M$$

μ-a.e. outside some arbitrary compact subset K of E with $K \supset K_0$ and

$$\widehat{A}_n(\phi) \le n \exp(\alpha C n^2), \ \phi(r) = \frac{Cr^2}{2}$$

for $n \gg 1$. Then $(T_t)_{t>0}$ is conservative.

8.2.1 Example one

We first consider a multi-dimensional example where a large variance compensates a strong drift.

Let $E = \mathbb{R}^2$ and $d\mu = \varphi dx$, where $\varphi = \xi^2$ with $\xi \in H^{1,2}_{loc}(\mathbb{R}^d, dx)$, $\varphi > 0$ dx-a.e. is such that

$$\varphi(x) = \frac{1}{5}|x|(|x|+1), \quad \mu\text{-a.e. } x \in K^c$$

where K is a compact subset of $\mathbb{R}^2$. Assume that $A = (a_{ij}) = (a_{ji}) \in H^{1,2}_{loc}(\mathbb{R}^2, \mu)$, $1 \le i, j \le 2$ is locally strictly elliptic (see (8.1)). Then the symmetric bilinear form

$$\mathcal{E}^0(f,g) := \int_{\mathbb{R}^2} \langle A(x)\nabla f(x), \nabla g(x)\rangle \mu(dx), \ f,g \in C_0^\infty(\mathbb{R}^2)$$

is closable on $L^2(\mathbb{R}^2, \mu)$. We further assume

$$|a_{11}(x)|, \ |a_{12}(x)| \le M_0(|x|+1)^2 \log(|x|+1),$$

μ-a.e. $x \in K^c$ for some constant $M_0 > 0$ and

$$a_{22}(x) = \frac{x_1^4}{x_2}, \quad \mu\text{-a.e. } x \in K^c$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. Because of the large $a_{22}(x)$, $(\mathcal{E}^0, D(\mathcal{E}^0))$ does not satisfy (7.5). Let $B(x) := \frac{1}{\varphi(x)}(x_2^2, -x_1^4)$. Then $B \in L^2_{loc}(\mathbb{R}^2, \mathbb{R}^2, \mu)$ and $|B| \in L^\infty_{loc}(K^c, \mu)$. Since

$$B(x) = \frac{1}{\varphi}(\partial_2 h, -\partial_1 h)$$

where $h(x) := \frac{1}{5}x_1^5 + \frac{1}{3}x_2^3 \in C^\infty(\mathbb{R}^2)$, we get

$$\int_{\mathbb{R}^2} \langle B, \nabla f \rangle d\mu = \int_{\mathbb{R}^2} (\partial_2 h \partial_1 f - \partial_1 h \partial_2 f) dx = 0, \text{ for any } f \in C_0^\infty(\mathbb{R}^2).$$

Then by the construction scheme of Section 8.1, we obtain a generalized Dirichlet form $\mathcal{E}$ given as an extension of

$$\int_{\mathbb{R}^2} \langle A(x)\nabla f(x), \nabla g(x)\rangle \mu(dx) - \int_{\mathbb{R}^2} \langle B(x), \nabla f(x)\rangle g(x)\mu(dx), \ f, g \in C_0^\infty(\mathbb{R}^2).$$

As we have seen in Section 8.1, (C1) and (C2) hold with $p = 1$, $Nv = \langle B, \nabla v\rangle$ and $D(N) = D(\mathcal{E}^0)_{loc,b}$. Let $\rho(x) := |x|$. Then $\rho \in D(\mathcal{E}^0)_{loc}$ and we obtain

$$\langle A\nabla\rho, \nabla\rho\rangle, \ \langle B, \nabla\rho\rangle \in L^\infty_{loc}(K^c, \mu),$$

hence (C3) holds. Since $C_0^\infty(\mathbb{R}^2) \subset D(L^0)_{0,b}$, (C) holds (i.e. (C4) holds). For $x \in K^c$, it holds

$$\begin{aligned} & \left| \langle A(x)\nabla|x|, \nabla|x|\rangle + \frac{(|x|+1)}{5}\langle B(x), \nabla|x|\rangle \right| \\ = \ & \frac{1}{|x|^2}\left| a_{11}(x)x_1^2 + 2a_{12}(x)x_1x_2 + a_{22}(x)x_2^2 + x_1x_2^2 - x_1^4x_2 \right| \\ \le \ & \frac{1}{|x|^2}\left| a_{11}(x)x_1^2 + 2a_{12}(x)x_1x_2 + x_1x_2^2 \right| \\ \le \ & M(|x|+1)^2\log(|x|+1) \end{aligned}$$

for some constant $M > 0$. Let

$$\phi(r) := 5\log(r+1),$$

i.e. $C = 5$ and $\beta = 1$ in Corollary 8.1(i). Then we obtain for $n \gg 1$, and some positive constant N

$$a_n \le Nn^4, \ b_n \le Nn^3, \ \|\langle B, \nabla\rho\rangle\|_{L^2(E_{4n}\setminus E_{2n}, \mu)} \le Nn^4,$$

which implies

$$\widehat{A}_n(\phi) \leq Nn^4.$$

Now choose $\alpha = \frac{4}{5}$ in Corollary 8.1(i) and obtain that $(T_t)_{t>0}$ is conservative.

8.2.2 Example two

Let $d = 1$ and $d\mu = \varphi dx$ where

$$\varphi(x) := \begin{cases} 1 & \text{if} \quad x > -1, \\ \dfrac{1}{|x|^3} & \text{if} \quad x \leq -1. \end{cases}$$

Then $\varphi \in L^1_{loc}(\mathbb{R}, dx)$ and μ is a σ-finite (not finite) measure on $\mathcal{B}(\mathbb{R})$. Let

$$A(x) := \begin{cases} (x + \sqrt{2})^2 & \text{if} \quad x \geq 0, \\ \dfrac{x^4 - x^3 + 6}{3} & \text{if} \quad x < 0. \end{cases}$$

Let $(\mathcal{E}^0, D(\mathcal{E}^0))$ be the symmetric Dirichlet form on $L^2(\mathbb{R}, \mu)$, which is the closure of $(\mathcal{E}^0, C_0^\infty(\mathbb{R}))$ on $L^2(\mathbb{R}, \mu)$ defined by

$$\mathcal{E}^0(f, g) := \int_{\mathbb{R}} A(x) f'(x) g'(x) \mu(dx), \;\; f, g \in C_0^\infty(\mathbb{R}).$$

Let d be the metric induced by Euclidean norm, i.e. $d(x, y) = |x - y|$. Put $\rho(x) := d(x, 0) = |x|$. Since $\Gamma(\rho, \rho)(x) = A(x)(\rho'(x))^2 = A(x)$, the first condition (7.5) in Proposition 7.1 can not hold. Let d^{int} be the intrinsic metric, i.e.

$$d^{int}(x, y) := \sup\left\{u(x) - u(y) : u \in D(\mathcal{E}^0)_{loc} \cap C(\mathbb{R}), \; \Gamma(u, u) \leq 1 \text{ on } \mathbb{R}\right\}.$$

Then, we obtain that

$$d^{int}(x, y) = \left| \int_x^y \frac{1}{\sqrt{A(z)}} dz \right|.$$

Indeed, let us fix $y \in \mathbb{R}$, then

$$u(x) := \left| \int_x^y \frac{1}{\sqrt{A(z)}} dz \right| \in D(\mathcal{E}^0)_{loc} \cap C(\mathbb{R})$$

satisfies $u(y) = 0$ and $\Gamma(u,u) = A \cdot (u')^2 = 1$. By definition of d^{int}, $d^{int}(x,y) \geq u(x)$. Suppose that $d^{int}(x,y) > u(x)$, then there exists $v \in D(\mathcal{E}^0)_{loc} \cap C(\mathbb{R})$ such that $\Gamma(v,v) \leq 1$ and $v(x) - v(y) > u(x) - u(y)$. However, $\Gamma(v,v) \leq 1$ implies that

$$-\frac{1}{\sqrt{A}} \leq v' \leq \frac{1}{\sqrt{A}}.$$

which further implies the contradiction $v(x) - v(y) \leq u(x) - u(y)$.

We have $\int_{-\infty}^0 \frac{1}{\sqrt{A(z)}} dz < \infty$, so $(-\infty, 0) \subset B_R^{d^{int}}$ for some $R > 0$. In other words, the ball $B_R^{d^{int}}$ induced by the metric d^{int} is not a relatively compact set in $\mathbb{R}$. Thus assumption (A) in [33] does not hold and we also can not apply [33, Theorem 4] to determine the conservativeness of $(\mathcal{E}^0, D(\mathcal{E}^0))$. However, by a scale function argument, we are able to show that $(\mathcal{E}^0, D(\mathcal{E}^0))$ is conservative. Indeed, since $A\varphi$ is continuous and strictly positive,

$$h(x) := \int_0^x \frac{1}{A(y)\varphi(y)} dy$$

is well-defined and satisfies

$$\mathcal{E}^0(h,g) = 0 \quad \text{for any } g \in C_0^\infty(\mathbb{R})$$

which implies that h is harmonic, i.e. $L^0 h = 0$. Thus we may regard h as canonical scale and $\frac{1}{h'A} dx = \varphi dx$ as the corresponding speed measure. Define

$$\Phi(x) := \int_0^x (h(x) - h(y)) \varphi(y) dy.$$

Then by Feller's test for non-explosion, $(\mathcal{E}^0, D(\mathcal{E}^0))$ is conservative, if and only if

$$\lim_{x \to \infty} \Phi(x) = \lim_{x \to -\infty} \Phi(x) = \infty.$$

If $x \geq 0$, then $\varphi(x) \equiv 1$ and

$$\Phi(x) = xh(x) - \int_0^x h(y)dy = \int_0^x h'(y)ydy = \int_0^x \frac{y}{(y+\sqrt{2})^2}dy \to \infty$$

as $x \to \infty$. In case $x < -1$, then

$$h(x) = c_1 + \int_{-1}^{x} \frac{-3y^3}{y^4 - y^3 + 6}dy = c_1 - \int_1^{-x} \frac{3y^3}{y^4 + y^3 + 6}dy$$

for some constant c_1, hence $h(x) \leq -\frac{3}{8}\log(-x) + c_1$ and $\lim_{x\to-\infty} h(x) = -\infty$. Furthermore, for $x < -1$

$$\begin{aligned}\Phi(x) = \int_0^x (h(x) - h(y))\varphi(y)dy &= h(x)\left(c_2 + \int_{-1}^{x} \varphi(y)dy\right) + c_3 - \int_{-1}^{x} h(y)\varphi(y)dy \\ &= h(x)\left(c_2 + \int_{-1}^{x} \frac{1}{-y^3}dy\right) + c_3 - \int_{-1}^{x} \frac{h(y)}{-y^3}dy \\ &\geq h(x)\left(c_2 + \frac{1}{2x^2} - \frac{1}{2}\right) + c_3 + \int_{-1}^{x} \frac{-3\log(-y) + 8c_1}{8y^3}dy\end{aligned}$$

where $c_2 < 0$, $c_3 > 0$ are some constants. Thus, $\lim_{x\to-\infty} \Phi(x) = \infty$. Consequently, $(\mathcal{E}^0, D(\mathcal{E}^0))$ is conservative.

Let $B(x) := \frac{1}{\varphi(x)}$. Then $|B| \in L^2_{loc}(\mathbb{R}, \mu)$ and satisfies

$$\int_{\mathbb{R}} B(x)f'(x)\mu(dx) = \int_{\mathbb{R}} f'(x)dx = 0$$

for any $f \in C_0^\infty(\mathbb{R})$. Consequently, by the construction scheme of Section 8.1, we can construct a generalized Dirichlet form $\mathcal{E}$ given as an extension of

$$\int_{\mathbb{R}} A(x)f'(x)g'(x)\mu(dx) - \int_{\mathbb{R}} B(x)f'(x)g(x)\mu(dx) \quad f, g \in C_0^\infty(\mathbb{R}).$$

Let $\rho(x) = |x|$. Then in the same way as in Example 8.2.1, we can obtain (C1)-(C4). If $x \geq 1$, then

$$\left|A(x) + \frac{(|x|+1)}{3}\left\langle B(x), \frac{x}{|x|}\right\rangle\right| \leq |(x+\sqrt{2})^2 + \frac{1}{3}(x+1)|$$

and if $x \le -1$, then

$$\left| A(x) + \frac{(|x|+1)}{3} \left\langle B(x), \frac{x}{|x|} \right\rangle \right| = 2.$$

Consequently,

$$\left| A(x) + \frac{(|x|+1)}{3} \left\langle B(x), \frac{x}{|x|} \right\rangle \right| \le M(|x|+1)^2,$$

where $M > 0$ is constant, i.e. $C = 3$, $\beta = 1$ and $\phi(r) := 3\log(r+1)$ in Corollary 8.1(i). Furthermore, for $n \gg 1$,

$$\widehat{A}_n(\phi) \le N n^{\frac{7}{2}}$$

where $N > 0$ is some constant. Now choose $\alpha := \frac{5}{6}$ in Corollary 8.1(i) and obtain that $(T_t)_{t>0}$ is conservative.

REMARK **8.1** *Since the above example is an example for a diffusion in $\mathbb{R}$, we are able to symmetrize $\mathcal{E}$ as done in Subsection 4.2.2, i.e. there is a symmetric Dirichlet form $(\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}}))$ in $L^2(\mathbb{R}, \widetilde{\mu})$ whose semigroup is locally equal to the semigroup $(T_t)_{t>0}$ of $\mathcal{E}$. Indeed, $(\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}}))$ can be expressed as the following form*

$$\widetilde{\mathcal{E}}(f,g) = \int_{\mathbb{R}} A(x) f'(x) g'(x) d\widetilde{\mu}$$

where $d\widetilde{\mu} = \widetilde{\varphi} dx$ and $\widetilde{\varphi}(x) = \exp\left(\int_0^x \frac{\varphi'}{\varphi}(s) + \frac{B}{A}(s) ds\right)$. By the same reason as for $(\mathcal{E}^0, D(\mathcal{E}^0))$ in the example above, we can not apply [33, Theorem 4] to determine the conservativeness of $\mathcal{E}$. However, by our results on the non-symmetric realization $\mathcal{E}$ of $\widetilde{\mathcal{E}}$ we obtain that $(\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}}))$ is conservative.

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국문초록

이 학위 논문에서 우리는 일반적인 거리 측도 공간에서 정의된 비대칭 디리클레 형식(Dirichlet form)의 비 부채꼴형(non-sectorial) 변화에 대한 재귀성(recurrence), 일시적임(transience), 보존성(conservativeness)의 해석적 기준에 대해 발전시켰다. 비대칭 디리클레 형식의 비 부채꼴형 변화는 [36] 에서 소개된 일반화된 디리클레 형식의 중요한 부류이다. 이 형식과 연관된 강한 Feller 과정(process)이 존재하는 경우, 이 해석적인 조건은 고전적 확률론적인 재귀성, 일시적임, 보존성(비폭발성)의 조건을 의미한다.

우리의 일반적인 결과의 응용으로, $\mathbb{R}^d$ 의 열린 또는 닫힌 부분집합에서 정의된 비대칭 부채꼴형 디리클레 형식 또는 대칭인 에너지 형식(energy form)의 1차 자유 발산의 변화인 일반화된 디리클레 형식을 고려했다. Carré du champ와 비 부채꼴형 1차 부분의 볼륨 증가 조건을 이용해서 우리는 재귀성과 보존성의 명백한 기준을 유도했다. 그리고 우리는 재귀성에 대한 응용으로 Muckenhoupt 가중치에 대한 구체적인 예제와 반례를 제시했다. 이 구체적인 반례는 비 부채꼴형이 대칭이나 비대칭 부채꼴형과 확연히 다르다는 것을 보여준다. 즉, 재귀성에 대한 기존의 기준이 일반화된 디리클레 형식에는 적용할 수 없다는 것을 말해준다. 게다가 우리는 보존성(비폭발성)에 대한 여러 가지 구체적인 예제를 제시했다. 이 예제들은 우리의 결과가 기존의 다른 여러 저자로부터 얻었던 보존성을 쉽게 얻을 수 있다는 것을 보여준다. 특별히, 드리프트가 변동(variance)을 상쇄 할 정도로 충분히 큰 경우에 3차의 변동에서도 보존성이 성립할 수 있다는 것을 보였다.

주요어: 일반화된 디리클레 형식, 비대칭 디리클레 형식, 재귀성, 일시적임, 보존성, 비폭발성, 마코스 반군, 확산과정.

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